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Generalized $\mathcal{W}_\infty$ higher-spin algebras and symbolic calculus on flag manifolds

Manuel Calixto*

Departamento de Matemática Aplicada y Estadística, Universidad Politécnica de Cartagena, Paseo Alfonso XIII 56, 30203 Cartagena, Spain

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Abstract

We study a new class of infinite-dimensional Lie algebras $\mathcal{W}_\infty(N_+, N_-)$ generalizing the standard $\mathcal{W}_\infty$ algebra, viewed as a tensor operator algebra of SU(1, 1) in a group-theoretic framework. Here we interpret $\mathcal{W}_\infty(N_+, N_-)$ either as an infinite continuation of the pseudo-unitary symmetry $U(N_+, N_-)$, or as a "higher-$U(N_+, N_-)$-spin extension" of the diffeomorphism algebra $\mathrm{diff}(N_+, N_-)$ of the $N = N_+ + N_-$ torus $U(1)^N$. We highlight this higher-spin structure of $\mathcal{W}_\infty(N_+, N_-)$ by developing the representation theory of $U(N_+, N_-)$ (discrete series), calculating higher-spin representations, coherent states and deriving Kähler structures on flag manifolds. They are essential ingredients to define operator symbols and to infer a geometric pathway between these generalized $\mathcal{W}_\infty$ symmetries and algebras of symbols of $U(N_+, N_-)$-tensor operators. Classical limits (Poisson brackets on flag manifolds) and quantum (Moyal) deformations are also discussed. As potential applications, we comment on the formulation of diffeomorphism-invariant gauge field theories, like gauge theories of higher-extended objects, and non-linear sigma models on flag manifolds.

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* Tel.: +34 968325587; fax: +34 968325694.
E-mail address: manuel.calixto@upct.es (M. Calixto).

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1. Introduction

The long sought-for unification of all interactions and exact solvability of (quantum) field theory and statistics parallels the quest for new symmetry principles. Symmetry is an essential resource when facing those two fundamental problems, either as a gauge guide principle or as a valuable classification tool. The representation theory of infinite-dimensional groups and algebras has not progressed very far, except for some important achievements in one and two dimensions (mainly Virasoro, $\mathcal{W}_\infty$ and Kac–Moody symmetries), and necessary breakthroughs in the subject remain to be carried out. The ultimate objective of this paper is to create a stepping stone to the development of a new class of infinite-dimensional symmetries, with potential useful applications in (quantum) field theory.

The structure of the proposed infinite symmetries resembles the one of the so-called $\mathcal{W}$ algebras. In the last decade, a large body of literature has been devoted to the study of $\mathcal{W}$-algebras, and the subject still continues to be fruitful. These algebras were first introduced as higher-conformal-spin $s > 2$ extensions [1] of the Virasoro algebra ($s = 2$) through the operator product expansion of the stress-energy tensor and primary fields in two-dimensional conformal field theory. $\mathcal{W}$-algebras have been widely used in two-dimensional physics, mainly in condensed matter, integrable models (Korteweg–de Vries, Toda), phase transitions in two dimensions, stringy black holes and, at a more fundamental level, as the underlying gauge symmetry of two-dimensional gravity models generalizing the Virasoro gauged symmetry in the light-cone discovered by Polyakov [2] by adding spin $s > 2$ currents (see e.g. [3–5] for a review). Only when all ($s \to \infty$) conformal spins $s \geq 2$ are considered, the algebra (denoted by $\mathcal{W}_\infty$) is proven to be of Lie type; moreover, currents of spin $s = 1$ can also be included [6], thus leading to the Lie algebra $\mathcal{W}_{1+\infty}$, which plays a determining role in the classification of all universality classes of incompressible quantum fluids and the identification of the quantum numbers of the excitations in the quantum Hall effect [7].

The process of elucidating the mathematical structure underlying $\mathcal{W}$ algebras has led to various directions. Geometric approaches identify the classical ($\hbar \to 0$) limit w_∞ of $\mathcal{W}_\infty$ algebras with area-preserving (symplectic) diffeomorphism algebras of two-dimensional surfaces [8,9]. These algebras possess a Poisson structure, and it is a current topic of great activity to recover the “quantum commutator” $[\cdot, \cdot]$ from (Moyal-like) deformations of the Poisson bracket $\{\cdot, \cdot\}$. There is a group-theoretic structure underlying these quantum deformations [10], according to which $\mathcal{W}_\infty$ algebras are just particular members of a one-parameter family $\mathcal{W}_\infty(c)$ of non-isomorphic [11,12] infinite-dimensional Lie-algebras of SU(1, 1) tensor operators (when “extended beyond the wedge” [10] or “analytically continued” [13]). The (field-theoretic) connection with the theory of higher-spin gauge fields in $(1 + 1)$- and $(2 + 1)$-dimensional anti-de Sitter space AdS [13–15] – homogeneous spaces of $SO(1, 2) \sim SU(1, 1)$ and $SO(2, 2) \sim SU(1, 1) \times SU(1, 1)$, respectively – is then apparent within this group-theoretical context. Also, the relationship between area-preserving diffeomorphisms and $\mathcal{W}_\infty$ algebras emerges naturally in this group-theoretic picture; indeed, it is well known that coadjoint orbits of any semisimple Lie group like $SU(1, 1) \simeq SL(2, \mathbb{R})$ (cone and hyperboloid of one and two sheets) naturally define a symplectic manifold, and the symplectic structure inherited from the group can be used to yield a Poisson bracket, which leads to a geometrical approach to quantization. From an algebraic point of view, the Poisson bracket is the classical limit of the quantum commutator of “covariant symbols” (see next section). However, the essence of the full quantum algebra is captured in a classical construction by extending the Poisson bracket to Moyal-like brackets. In particular, one can reformulate the (cumbersome) problem of calculating commutators of tensor operators of su(1, 1) in terms of (easier to perform) Moyal (deformed) brackets of polynomial functions on coadjoint orbits $\mathbb{O}$ of SU(1, 1). A further simplification, that we shall use, then consists of taking advantage of the standard oscillator realization (2.4) of the semisimple Lie algebra generators and replacing non-canonical (2.12) by Heisenberg brackets (4.5).

Going from three-dimensional algebras su(2) and su(1, 1) to higher-dimensional pseudo-unitary algebras su(N_+, N_-) entails non-trivial problems. Actually, the classification and labelling of tensor operators of Lie groups other than SU(1, 1) and SU(2) is not an easy task in general. In the letter [16], the author put forth an infinite set $\mathcal{W}_\infty(N_+, N_-)$ of tensor operators of $U(N_+, N_-)$ and calculated the structure constants of this quantum associative operator algebra by taking advantage of the oscillator realization of the $U(N_+, N_-)$ Lie-algebra, in terms of $N = N_+ + N_-$ boson operators [see Eq. (2.4)], and by using Moyal brackets. Operator labelling coincides here with the standard Gel'fand–Weyl pattern for vectors in the carrier space of unirreps of $U(N)$ (see later on Section 5.2). Later on, the particular case of $\mathcal{W}_\infty(2, 2)$ was identified in [17] with a four-dimensional analogue of the Virasoro algebra, i.e. an infinite extension ("promotion or analytic continuation" in the sense of [13]) of the finite-dimensional conformal symmetry SU(2, 2) $\sim$ SO(4, 2) in $(3 + 1)$ dimensions. Also, $\mathcal{W}_\infty(2, 2)$ was interpreted as a higher-conformal-spin extension of the diffeomorphism algebra diff(4) of vector fields on a four-dimensional manifold (just as $\mathcal{W}_\infty$ is a higher-spin extension of the Virasoro diff(1) algebra), thus constituting a potential gauge guide principle towards the formulation of induced conformal gravities (Wess–Zumino–Witten-like models) in realistic dimensions [18]. For completeness, let us say that $\mathcal{W}_\infty$-algebras also appear as central extensions of the algebra of (pseudo-)differential operators on the circle [19], and higher-dimensional analogues have been constructed in that context [20]; however, we do not find a clear connection with our construction.

In this article the aim is to infer a concrete pathway between these natural (algebraic) generalizations $\mathcal{W}_\infty(N_+, N_-)$ of $\mathcal{W}_\infty$, and infinite higher-spin algebras of $U(N_+, N_-)$ operator symbols, using the coherent-state machinery and tools of geometric and Berezin quantization. In order to justify the view of $\mathcal{W}_\infty(N_+, N_-)$ as a "higher-spin algebra" of $U(N_+, N_-)$, we shall develop the representation theory of $U(N_+, N_-)$, calculating arbitrary-spin coherent states and deriving Kähler structures on flag manifolds, which are essential ingredients to define operator symbols, star-products and to compute the leading order ($\hbar \to 0$, or large quantum numbers) structure constants of star-commutators in terms of Poisson brackets on the flag space. Actually, the structure constants calculated in [16] were restricted to a class of irreducible representations given by oscillator representations. Here we show how to deal with the general case.

Throughout the paper, we shall discuss either classical limits of quantum structures (Poisson brackets from star-commutators) or quantum deformations of classical objects (Moyal deformations of oscillator algebras).

We believe this paper touches a wide range of different algebraic and geometric structures of importance in Physics and Mathematics. Our main objective is to describe them and to propose interconnections between them. Therefore, except for Section 2, which summarizes some basic definitions and theorems found in the literature, we have rather preferred to follow a fairly descriptive approach throughout the paper. Perhaps pure Mathematicians will miss the "Theorem-Proof" procedure to present some of the particular results of this work, but I hope our plan will make the presentation more dynamic and will result in greater dissemination of the underlying ideas and methods.

The organization of the paper is as follows. Firstly we set the general context of our problem and remind some basic theorems and notions on the representation theory of Lie groups (in particular, we focus on pseudo-unitary groups) and geometric structures derived from it. In Section 3 we exemplify the previous structural information with the case of three-dimensional underlying algebras su(2) and su(1, 1), their tensor operator algebras, classical limits, Lie–Poisson structures and their relevance in large-N matrix models (and relativistic membranes) and $\mathcal{W}_{(1+)\infty}$ invariant theories. In Section 4 we extend these constructions to general pseudo-unitary groups and we show how to build "generalized w_∞ algebras" $w_\infty(N_+, N_-)$ and to compute their quantum (Moyal) deformations $\mathcal{W}_\infty(N_+, N_-)$ through oscillator realizations of the $u(N_+, N_-)$ Lie algebra. Then, in Section 5 we introduce a local complex parametrization of the coset representatives $\mathrm{SU}(N)/U(1)^N = \mathbb{F}_{N-1}$ (flag space), we construct coherent states and derive Kähler structures on flag manifolds. They are essential ingredients to discuss symbolic calculus on flag manifolds, and to highlight the higher-spin structure of the algebra $\mathcal{W}_\infty(N_+, N_-)$. In Section 6 we make some comments on the potential role of these infinite-dimensional algebras as residual gauge symmetries of extended objects ("$N(N-1)$-branes $\mathbb{F}_{N-1}$") in the light-cone gauge, and formulate non-linear sigma models on flag manifolds. Section 7 is devoted to conclusions and outlook.

2. The group-theoretical backdrop

Let us start by fixing notation and reminding some definitions and results on group, tensor operator, Poisson–Lie algebras, coherent states and symbols of a Lie group G; in particular, we shall focus on pseudo-unitary groups:

$$G = U(N_+, N_-) = \{g \in M_{N\times N}(\mathbb{C})/, g\Lambda g^\dagger = \Lambda\}, \quad N = N_+ + N_-, \tag{2.1}$$

that is, groups of complex $N \times N$ matrices g that leave invariant the indefinite metric $\Lambda = \mathrm{diag}(1, \ldots^{N_+}, 1, -1, \ldots^{N_-}, -1)$. The Lie-algebra $\mathcal{G}$ is generated by the step operators $\hat{X}_\alpha^\beta$,

$$\mathcal{G} = u(N_+, N_-) = \langle \hat{X}_\alpha^\beta, \text{with } (\hat{X}_\alpha^\beta)_\mu^\nu \equiv \hbar\delta_\alpha^\nu\delta_\mu^\beta; \alpha, \beta, \mu, \nu = 1, \ldots, N\rangle, \tag{2.2}$$

(we introduce the Planck constant $\hbar$ for convenience) with commutation relations:

$$[\hat{X}_{\alpha_1}^{\beta_1}, \hat{X}_{\alpha_2}^{\beta_2}] = \hbar(\delta_{\alpha_2}^{\beta_1}\hat{X}_{\alpha_1}^{\beta_2} - \delta_{\alpha_1}^{\beta_2}\hat{X}_{\alpha_2}^{\beta_1}). \tag{2.3}$$

There is a standard *oscillator realization* of these step operators in terms of N boson operator variables $(\hat{a}_\alpha^\dagger, \hat{a}^\beta)$, given by:

$$\hat{X}_\alpha^\beta = \hat{a}_\alpha^\dagger\hat{a}^\beta, \qquad [\hat{a}^\beta, \hat{a}_\alpha^\dagger] = \hbar\delta_\alpha^\beta\mathbb{I}, \quad \alpha, \beta = 1, \ldots N, \tag{2.4}$$

which reproduces (2.3) (we use the metric Λ to raise and lower indices). Thus, for unitary irreducible representations of $U(N_+, N_-)$ we have the conjugation relation:

$$(\hat{X}^{\beta}_{\alpha})^{\dagger} = \Lambda^{\beta\mu}\hat{X}^{\nu}_{\mu}\Lambda_{\nu\alpha} \tag{2.5}$$

(sum over doubly occurring indices is understood unless otherwise stated). Sometimes it will be more convenient to use the generators $\hat{X}_{\alpha\beta} = \Lambda_{\alpha\mu}\hat{X}^{\mu}_{\beta}$ instead of $\hat{X}^{\beta}_{\alpha}$, for which the conjugation relation (2.5) is simply written as $\hat{X}^{\dagger}_{\alpha\beta} = \hat{X}_{\beta\alpha}$, and the commutation relations (2.3) adopt the form:

$$[\hat{X}_{\alpha_1\beta_1}, \hat{X}_{\alpha_2\beta_2}] = \hbar(\Lambda_{\alpha_2\beta_1}\hat{X}_{\alpha_1\beta_2} - \Lambda_{\alpha_1\beta_2}\hat{X}_{\alpha_2\beta_1}). \tag{2.6}$$

The oscillator realization (2.4) of $u(N_+, N_-)$-generators will be suitable for our purposes later on.

Definition 2.1. Let $\mathcal{G}^{\otimes}$ be the tensor algebra over $\mathcal{G}$, and $\mathcal{I}$ the ideal of $\mathcal{G}^{\otimes}$ generated by $[\hat{X}, \hat{Y}] - (\hat{X}\otimes\hat{Y} - \hat{Y}\otimes\hat{X})$, where $\hat{X}, \hat{Y} \in \mathcal{G}$. The universal enveloping algebra $\mathcal{U}(\mathcal{G})$ is the quotient $\mathcal{G}^{\otimes}/\mathcal{I}$.

[From now on we shall drop the $\otimes$ symbol in writing tensor products.]

Theorem 2.2 (Poincaré–Birkhoff–Witt). *The monomials* $\hat{X}^{k_1}_{\alpha_1\beta_1}\cdots\hat{X}^{k_n}_{\alpha_n\beta_n}$, *with* $k_i \geq 0$, *form a basis of* $\mathcal{U}(\mathcal{G})$.

Casimir operators are especial elements of $\mathcal{U}(\mathcal{G})$, which commute with everything. There are N Casimir operators for $U(N_+, N_-)$, which are written as polynomials of degree $1, 2, \ldots, N$ of step operators as follows:

$$\hat{C}_1 = \hat{X}^{\alpha}_{\alpha}, \qquad \hat{C}_2 = \hat{X}^{\beta}_{\alpha}\hat{X}^{\alpha}_{\beta}, \qquad \hat{C}_3 = \hat{X}^{\beta}_{\alpha}\hat{X}^{\gamma}_{\beta}\hat{X}^{\alpha}_{\gamma}, \ldots. \tag{2.7}$$

The universal enveloping algebra $\mathcal{U}(\mathcal{G})$ decomposes into *factor or quotient Lie algebras* $\mathcal{W}_c(\mathcal{G})$, with $c = (c_1, \ldots, c_N)$ an arbitrary N-dimensional complex vector, as follows. Let

$$\mathcal{I}_c = \sum_{\alpha=1}^{N}(\hat{C}_\alpha - \hbar^{\alpha}c_\alpha)\mathcal{U}(\mathcal{G})$$

be the ideal generated by the Casimir operators $\hat{C}_\alpha$. The quotient $\mathcal{W}_c(\mathcal{G}) \equiv \mathcal{U}(\mathcal{G})/\mathcal{I}_c$ is a Lie algebra. Roughly speaking, this quotient means that we replace $\hat{C}_\alpha$ by the complex c-number $C_\alpha \equiv \hbar^{\alpha}c_\alpha$ whenever it appears in the commutators of elements of $\mathcal{U}(\mathcal{G})$.

Definition 2.3. We shall refer to $\mathcal{W}_c(\mathcal{G})$ as a c-tensor operator algebra.

According to Burnside's theorem [21], for some critical values $c_\alpha = c^{(0)}_\alpha$, the infinite-dimensional Lie algebra $\mathcal{W}_c(\mathcal{G})$ "collapses" to a finite-dimensional one. In a more formal language:

Theorem 2.4 (Burnside). *When $c_\alpha, \alpha = 1, \ldots, N$ coincide with the eigenvalues of $\hat{C}_\alpha$ in a d_c-dimensional irrep D_c* of G, *there exists an ideal $\chi \subset \mathcal{W}_c(\mathcal{G})$ such that $\mathcal{W}_c(\mathcal{G})/\chi =$* $\mathrm{sl}(d_c, \mathbb{C})$, *or* $\mathrm{su}(d_c)$, *by taking a compact real form of the complex Lie algebra.*

Another interesting structure related to the previous one is the *group C^*-algebra* $\mathbb{C}^*(G)$ [in order to avoid some technical difficulties, let us restrict ourselves to the compact G case in the next discussion]:

Definition 2.5. Let $C^\infty(G)$ be the set of analytic complex functions Ψ on G,

$$C^\infty(G) = \{\Psi : G \to \mathbb{C}, g \mapsto \Psi(g)\}. \tag{2.8}$$

The group algebra $\mathbb{C}^*(G)$ is a C^*-algebra with an invariant associative $*$-product (convolution product)

$$(\Psi * \Psi')(g') \equiv \int_G \mathrm{d}^L g \Psi(g)\Psi'(g^{-1} \bullet g'), \tag{2.9}$$

($g \bullet g'$ denotes the composition group law and $\mathrm{d}^L g$ stands for the left Haar measure) and an involution $\Psi^*(g) \equiv \bar{\Psi}(g^{-1})$.

The conjugate space $R(G)$ of $C^\infty(G)$ consists of all generalized functions with compact supports. The space $M_0(G)$ of all regular Borel measures with compact support is a subspace of $R(G)$. The set $R(G, H)$ of all generalized functions on G with compact supports contained in a subgroup H also forms a subspace of $R(G)$. The following theorem (see [21]) reveals a connection between $R(G, \{e\})$ [$e \in G$ denotes the identity element] and the enveloping algebra:

Theorem 2.6 (L. Schwartz). *The algebra $R(G, \{e\})$ is isomorphic to the enveloping algebra $\mathcal{U}(\mathcal{G})$.*

This isomorphism is apparent when we realize the Lie algebra $\mathcal{G}$ by left invariant vector fields $\hat{X}^L$ on G and consider the mapping $\Phi : \mathcal{G} \to R(G)$, $\hat{X} \mapsto \Phi_{\hat{X}}$, defined by the formula

$$\langle \Phi_{\hat{X}} | \Psi \rangle \equiv (\hat{X}^L \Psi)(e) \quad \forall \Psi \in C^\infty(G), \tag{2.10}$$

where $\langle \Phi | \Psi \rangle \equiv \int_G \mathrm{d}^L g\, \bar{\Phi}(g)\Psi(g)$ denotes a scalar product and $(\hat{X}^L \Psi)(e)$ means the action of $\hat{X}^L$ on Ψ restricted to the identity element $e \in G$. One can also verify the relation

$$\langle \Phi_{\hat{X}_1} * \cdots * \Phi_{\hat{X}_n} | \Psi \rangle = (\hat{X}_1^L \cdots \hat{X}_n^L \Psi)(e) \quad \forall \Psi \in C^\infty(G), \tag{2.11}$$

between star products in $R(G)$ and tensor products in $\mathcal{U}(\mathcal{G})$:

Let us comment now on the geometric counterpart of the previous algebraic structures, by using the language of geometric quantization.

The classical limit of the convolution commutator $[\Psi, \Psi'] = \Psi * \Psi' - \Psi' * \Psi$ corresponds to the Poisson–Lie bracket

$$\{\psi, \psi'\}_{\mathrm{PL}}(g) = \lim_{\hbar \to 0} \frac{\mathrm{i}}{\hbar^2} [\Psi, \Psi'](g) = \mathrm{i}(\Lambda_{\alpha_2\beta_1} x_{\alpha_1\beta_2} - \Lambda_{\alpha_1\beta_2} x_{\alpha_2\beta_1}) \frac{\partial \psi}{\partial x_{\alpha_1\beta_1}} \frac{\partial \psi'}{\partial x_{\alpha_2\beta_2}} \tag{2.12}$$

between smooth functions $\psi \in C^\infty(\mathcal{G}^*)$ on the coalgebra $\mathcal{G}^*$, where $x_{\alpha\beta}, \alpha, \beta = 1, \ldots, N$ denote a coordinate system in the coalgebra $\mathcal{G}^* = u(N_+, N_-)^* \simeq \mathbb{R}^{N^2}$, seen as a N^2-dimensional vector space. The "quantization map" relating Ψ and ψ is symbolically given by the expression:

$$\Psi(g) = \int_{\mathcal{G}^*} \frac{\mathrm{d}^{N^2}\Theta}{(2\pi\hbar)^{N^2}}\, \mathrm{e}^{(i/\hbar)\Theta(\hat{X})}\psi(\Theta), \tag{2.13}$$

where $g = \exp(\hat{X}) = \exp(x^{\alpha\beta}\hat{X}_{\alpha\beta})$ is an element of G and $\Theta = \theta_{\alpha\beta}\Theta^{\alpha\beta}$ is an element of $\mathcal{G}^*$.

The constraints $\hat{C}_\alpha(x) = C_\alpha = \hbar^\alpha c_\alpha$ defined by the Casimir operators (2.7) (written in terms of the coordinates $x_{\alpha\beta}$ instead of $\hat{X}_{\alpha\beta}$) induce a foliation

$$\mathcal{G}^* \simeq \bigcup_C \mathbb{O}_C \tag{2.14}$$

of the coalgebra $\mathcal{G}^*$ into leaves $\mathbb{O}_C$: coadjoint orbits, algebraic (flag) manifolds (see later on Section 5). This foliation is the (classical) analogue of the (quantum) standard Peter–Weyl decomposition (see [22]) of the group algebra $\mathbb{C}^*(G)$:

Theorem 2.7 (Peter–Weyl). *Let G be a compact Lie group. The group algebra $\mathbb{C}^*(G)$ decomposes,*

$$\mathbb{C}^*(G) \simeq \bigoplus_{c\in\hat{G}} \mathcal{W}_c(\mathcal{G}), \tag{2.15}$$

into factor algebras $\mathcal{W}_c(\mathcal{G})$, where $\hat{G}$ denotes the space of all (equivalence classes of) irreducible representations of G of dimension d_c.

The leaves $\mathbb{O}_C$ admit a symplectic structure $(\mathbb{O}_C, \Omega_C)$, where Ω_C denotes a closed 2-form (a Kähler form), which can be obtained from a Kähler potential K_C as:

$$\Omega_C(z, \bar{z}) = \frac{\partial^2 K_C(z, \bar{z})}{\partial z_{\alpha\beta}\partial \bar{z}_{\sigma\nu}}\, \mathrm{d}z_{\alpha\beta} \wedge \mathrm{d}\bar{z}_{\sigma\nu} = \Omega_C^{\alpha\beta;\sigma\nu}(z, \bar{z})\, \mathrm{d}z_{\alpha\beta} \wedge \mathrm{d}\bar{z}_{\sigma\nu}, \tag{2.16}$$

where $z_{\alpha\beta}, \alpha > \beta$ denotes a system of complex coordinates in $\mathbb{O}_C$ (see later on Section 5.1).

After the foliation of $C^\infty(\mathcal{G}^*)$ into Poisson algebras $C^\infty(\mathbb{O}_C)$, the Poisson bracket induced on the leaves $\mathbb{O}_C$ becomes:

$$\{\psi_l^c, \psi_m^c\}_P(z, \bar{z}) = \sum_{\alpha_j > \beta_j} \Omega^C_{\alpha_1\beta_1;\alpha_2\beta_2}(z, \bar{z}) \frac{\partial \psi_l^c(z, \bar{z})}{\partial z_{\alpha_1\beta_1}} \frac{\partial \psi_m^c(z, \bar{z})}{\partial \bar{z}_{\alpha_2\beta_2}} = \sum_n f_{lm}^n(c)\psi_n^c(z, \bar{z}). \tag{2.17}$$

The structure constants for (2.17) can be obtained through the scalar product $f_{lm}^n(c) = \langle \psi_n^c | \{\psi_l^c, \psi_m^c\}_P\rangle$, with integration measure (2.18), when the set $\{\psi_n^c\}$ is chosen to be orthonormal.

To each function $\psi \in C^\infty(\mathbb{O}_C)$, one can assign its Hamiltonian vector field $H_\psi \equiv \{\psi, \cdot\}_P$, which is divergence-free and preserves de natural volume form

$$\mathrm{d}\mu_C(z,\bar{z}) = (-1)^{\binom{n}{2}} \frac{1}{n!} \Omega_C^n(z,\bar{z}), \quad 2n = \dim(\mathbb{O}_C). \tag{2.18}$$

In general, any vector field H obeying $L_H\Omega = 0$ (with $L_H \equiv i_H \circ d + d \circ i_H$ the Lie derivative) is called locally Hamiltonian. The space $\mathrm{LHam}(\mathbb{O})$ of locally Hamiltonian vector fields is a subalgebra of the algebra $\mathrm{sdiff}(\mathbb{O})$ of symplectic (volume-preserving) diffeomorphisms of $\mathbb{O}$, and the space $\mathrm{Ham}(\mathbb{O})$ of Hamiltonian vector fields is an ideal of $\mathrm{LHam}(\mathbb{O})$. The two-dimensional case $\dim(\mathbb{O}) = 2$ is special because $\mathrm{sdiff}(\mathbb{O}) = \mathrm{LHam}(\mathbb{O})$, and the quotient $\mathrm{LHam}(\mathbb{O})/\mathrm{Ham}(\mathbb{O})$ can be identified with the first de-Rham cohomology class $H^1(\mathbb{O},\mathbb{R})$ of $\mathbb{O}$ via $H \mapsto i_H\Omega$.

Poisson and symplectic diffeomorphism algebras of $\mathbb{O}_{C_+} = S^2$ and $\mathbb{O}_{C_-} = S^{1,1}$ (the sphere and the hyperboloid) appear as the classical limit [small $\hbar$ and large (conformal-) spin $c_\pm = s(s \pm 1)$, so that the curvature radius $C_\pm = \hbar^2 c_\pm$ remains finite]:

$$\begin{aligned}
&\lim_{\substack{c_+\to\infty\\ \hbar\to 0}} \mathcal{W}_{c_+}(\mathrm{su}(2)) \simeq C^\infty(S^2) \simeq \mathrm{sdiff}(S^2) \simeq \mathrm{su}(\infty),\\
&\lim_{\substack{c_-\to\infty\\ \hbar\to 0}} \mathcal{W}_{c_-}(\mathrm{su}(1,1)) \simeq C^\infty(S^{1,1}) \simeq \mathrm{sdiff}(S^{1,1}) \simeq \mathrm{su}(\infty,\infty)
\end{aligned} \tag{2.19}$$

of factor algebras of SU(2) and SU(1, 1), respectively (see [13,12]).[1]

Let us clarify the classical limits (2.19) by making use of the *operator (covariant) symbols* [24]:

$$L^c(z,\bar{z}) \equiv \langle cz|\hat{L}|cz\rangle, \qquad \hat{L} \in \mathcal{W}_c(\mathcal{G}), \tag{2.20}$$

constructed as the mean value of an operator $\hat{L} \in \mathcal{W}_c(\mathcal{G})$ in the coherent state $|cz\rangle$ (see later on Section 5.2 for more details). Using the resolution of unity:

$$\int_{\mathbb{O}_C} |cu\rangle\langle cu|\, \mathrm{d}\mu_C(u,\bar{u}) = 1 \tag{2.21}$$

for coherent states, one can define the so-called *star multiplication of symbols* $L_1^c \star L_2^c$ as the symbol of the product $\hat{L}_1\hat{L}_2$ of two operators $\hat{L}_1$ and $\hat{L}_2$:

$$(L_1^c \star L_2^c)(z,\bar{z}) \equiv \langle cz|\hat{L}_1\hat{L}_2|cz\rangle = \int_{\mathbb{O}_C} L_1^c(z,\bar{u})L_2^c(u,\bar{z})\, \mathrm{e}^{-s_c^2(z,u)}\, \mathrm{d}\mu_c(u,\bar{u}), \tag{2.22}$$

where we introduce the non-diagonal symbols

$$L^c(z,\bar{u}) = \frac{\langle cz|\hat{L}|cu\rangle}{\langle cz|cu\rangle}, \tag{2.23}$$

[1] The approximation $\mathrm{sdiff}(S^2) \simeq \mathrm{su}(\infty)$ is still not well understood and additional work should be done towards its satisfactory formulation. In [23] the approach to approximate $\mathrm{sdiff}(S^2)$ and $\mathrm{sdiff}(T^2)$ by $\lim_{N\to\infty} \mathrm{su}(N)$ was studied and a weak uniqueness theorem was proved; however, whether choices of sets of basis functions on spaces with different topologies do in fact correspond to distinct algebras deserves more careful study.

and $s_c^2(z,u) \equiv -\ln|\langle cz|cu\rangle|^2$ can be interpreted as the square of the distance between the points z, u on the coadjoint orbit $\mathbb{O}_C$. Using general properties of coherent states [25], it can be easily seen that $s_c^2(z,u) \geq 0$ tends to infinity with $c \to \infty$, if $z \neq u$, and equals zero if $z = u$. Thus, one can conclude that, in that limit, the domain $u \approx z$ gives only a contribution to the integral (2.22). Decomposing the integrand near the point $u = z$ and going to the integration over $w = u - z$, it can be seen that the Poisson bracket (2.17) provides the first order approximation to the star commutator for large quantum numbers c (small $\hbar$); that is:

$$L_1^c \star L_2^c - L_2^c \star L_1^c = i\{L_1^c, L_2^c\}_P + \mathrm{O}(1/c_\alpha), \tag{2.24}$$

i.e. the quantities $1/c_\alpha \sim \hbar^\alpha$ (inverse Casimir eigenvalues) play the role of the Planck constant $\hbar$, and one uses that $\mathrm{d}s_c^2 = \Omega_C^{\alpha\beta;\sigma\nu}\,\mathrm{d}z_{\alpha\beta}\,\mathrm{d}\bar{z}_{\sigma\nu}$ (Hermitian Riemannian metric on $\mathbb{O}_C$). We address the reader to Section 5 for more details.

Before going to the general $\mathrm{SU}(N_+, N_-)$ case, let us discuss the two well known examples of SU(2) and SU(1, 1).

3. Tensor operator algebras of SU(2) and SU(1, 1)

3.1. Tensor operator algebras of SU(2) and large-N matrix models

Let $\hat{J}_i^{(N)}, i = 1, 2, 3$ be three $N \times N$ Hermitian matrices with commutation relations:

$$[\hat{J}_i^{(N)}, \hat{J}_j^{(N)}] = \mathrm{i}\hbar\epsilon_{ijk}\hat{J}_k^{(N)}, \tag{3.1}$$

that is, a N-dimensional irreducible representation of the angular momentum algebra su(2). The Casimir operator $\hat{C}_2 = (\hat{J}^{(N)})^2 = \hbar^2 \frac{N^2-1}{4}\mathbb{I}_{N\times N}$ is a multiple of the $N \times N$ identity matrix $\mathbb{I}$. The factor algebra $\mathcal{W}_N(\mathrm{su}(2))$ is generated by the SU(2)-tensor operators:

$$\hat{T}_m^I(N) \equiv \sum_{\substack{i_k=1,2,3\\ k=1,\dots,I}} \kappa_{i_1,\cdots,i_I}^{(m)} \hat{J}_{i_1}^{(N)} \cdots \hat{J}_{i_I}^{(N)}, \tag{3.2}$$

where the upper index $I = 1, \dots, N-1$ is the spin label, $m = -I, \dots, I$ is the third component and the complex coefficients $\kappa_{i_1,\cdots,i_I}^{(m)}$ are the components of a symmetric and traceless tensor. According to Burnside's Theorem 2.4, the factor algebra $\mathcal{W}_N(\mathrm{su}(2))$ is isomorphic to $\mathrm{su}(N)$. Thus, the commutation relations:

$$[\hat{T}_m^I(N), \hat{T}_n^J(N)] = f_{mnK}^{IJl}(N)\hat{T}_l^K(N) \tag{3.3}$$

are those of the $\mathrm{su}(N)$ Lie algebra, where $f_{mnK}^{IJl}(N)$ symbolize the structure constants which, for the Racah–Wigner basis of tensor operators [26], can be written in terms of Clebsch–Gordan and (generalized) $6j$-symbols [27,10,13].

The formal limit $N \to \infty$ of the commutation relations (3.3) coincides with the Poisson bracket

$$\{Y_m^I, Y_n^J\}_P = \frac{\mathrm{i}}{\sin\vartheta}\left(\frac{\partial Y_m^I}{\partial\vartheta}\frac{\partial Y_n^J}{\partial\varphi} - \frac{\partial Y_m^I}{\partial\varphi}\frac{\partial Y_n^J}{\partial\vartheta}\right) = f_{mnK}^{IJl}(\infty)Y_l^K \tag{3.4}$$

between spherical harmonics

$$Y_m^I(\vartheta,\varphi) \equiv \sum_{\substack{i_k=1,2,3\\ k=1,\dots,I}} \kappa^{(m)}_{i_1,\dots,i_I} x_{i_1}\cdots x_{i_I}, \tag{3.5}$$

which are defined in a similar way to tensor operators (3.2), but replacing the angular momentum operators $\hat{J}^{(N)}$ by the coordinates $x = (\cos\varphi\sin\vartheta, \sin\varphi\sin\vartheta, \cos\vartheta)$, i.e. its covariant symbols (2.20). Indeed, the large-N structure constants can be calculated through the scalar product (see [28]):

$$\begin{aligned}\lim_{N\to\infty} f^{IJl}_{mnK}(N) &= f^{IJl}_{mnK}(\infty) = \langle Y_l^K | \{Y_m^I, Y_n^J\}_P\rangle \\ &= \int_{S^2} \sin\vartheta\, \mathrm{d}\vartheta\, \mathrm{d}\varphi \bar{Y}_l^K(\vartheta,\varphi)\{Y_m^I, Y_n^J\}_P(\vartheta,\varphi).\end{aligned}$$

The set of Hamiltonian vector fields $H_m^I \equiv \left\{Y_m^I, \cdot\right\}_P$ close the algebra sdiff(S^2) of area-preserving diffeomorphisms of the sphere, which can be identified with su(∞) in the ("weak convergence") sense of [23]—see Eq. (2.19). This fact was used in [27] to approximate the residual gauge symmetry sdiff(S^2) of the relativistic spherical membrane by su$(N)|_{N\to\infty}$. There is an intriguing connection between this theory and the quantum mechanics of space constant ("vacuum configurations") SU(N) Yang–Mills potentials

$$A_\mu(x)^i_j = \sum_{a=1}^{N^2-1} A^a_\mu(x)(\hat{T}_a)^i_j, \quad \hat{T}_a = \hat{T}^I_m(N),\ a = 1,\dots,N^2-1 \tag{3.6}$$

in the limit of "large number of colours" (large-N). Indeed, the low-energy limit of the SU(∞) Yang–Mills action

$$\begin{aligned}&\mathcal{S} = \int \mathrm{d}^4x \langle F_{\mu\nu}(x)|F^{\mu\nu}(x)\rangle, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \{A_\mu, A_\nu\}_P, \\ &A_\mu(x;\vartheta,\varphi) = \sum_{I,m} A^{Im}_\mu(x) Y^I_m(\vartheta,\varphi),\end{aligned} \tag{3.7}$$

described by space-constant SU(∞) vector potentials $X_\mu(\tau;\vartheta,\varphi) \equiv A_\mu(\tau,\vec{0};\vartheta,\varphi)$, turns out to reproduce the dynamics of the relativistic spherical membrane (see [28]). Moreover, space–time constant SU(∞) vector potentials $X_\mu(\vartheta,\varphi) \equiv A_\mu(0;\vartheta,\varphi)$ lead to the Schild action density for (null) strings [29]; the argument that the internal symmetry space of the $U(\infty)$ pure Yang–Mills theory must be a functional space, actually the space of configurations of a string, was pointed out in Ref. [30]. Replacing the Sdiff(S^2)-gauge invariant theory (3.7) by a SU(N)-gauge invariant theory with vector potentials (3.6) then provides a form of regularization.

We shall see later in Section 6.1 how actions for relativistic symplectic p-branes (higher-dimensional coadjoint orbits) can be defined for general (pseudo-)unitary groups in a similar way.

3.2. Tensor operator algebras of SU(1,1) and $\mathcal{W}_{(1+)\infty}$ symmetry

As already stated in Section 1, $\mathcal{W}$ algebras were first introduced as higher-conformal-spin ($s > 2$) extensions [1] of the Virasoro algebra ($s = 2$) through the operator product expansion of the stress-energy tensor and primary fields in two-dimensional conformal field theory. Only when all ($s \to \infty$) conformal spins are considered, the algebra (denoted by $\mathcal{W}_\infty$) is proven to be of Lie type.

Their classical limit w proves to have a space–time origin as (symplectic) diffeomorphism algebras and Poisson algebras of functions on symplectic manifolds. For example, $w_{1+\infty}$ is related to the algebra of area-preserving diffeomorphisms of the cylinder. Actually, let us choose the next set of classical functions of the bosonic (harmonic oscillator) variables $a(\bar{a}) = \frac{1}{\sqrt{2}}(q \pm \mathrm{i}p) = \rho\,\mathrm{e}^{\pm\mathrm{i}\vartheta}$ (we are using mass and frequency $m = 1 = \omega$, for simplicity):

$$L^I_{+|n|} \equiv \tfrac{1}{2}(a\bar{a})^{I-|n|}a^{2|n|} = \tfrac{1}{2}\rho^{2I}\,\mathrm{e}^{2\mathrm{i}|n|\vartheta},$$
$$L^I_{-|n|} \equiv \tfrac{1}{2}(a\bar{a})^{I-|n|}\bar{a}^{2|n|} = \tfrac{1}{2}\rho^{2I}\,\mathrm{e}^{-2\mathrm{i}|n|\vartheta}, \tag{3.8}$$

where $n \in \mathbb{Z}$; $I \in \mathbb{Z}^+$. A straightforward calculation from the basic Poisson bracket $\{a, \bar{a}\} = \mathrm{i}$ provides the following formal Poisson algebra:

$$\{L^I_m, L^J_n\} = \mathrm{i}\left(\frac{\partial L^I_m}{\partial a}\frac{\partial L^J_n}{\partial \bar{a}} - \frac{\partial L^I_m}{\partial \bar{a}}\frac{\partial L^J_n}{\partial a}\right) = \mathrm{i}(In - Jm)L^{I+J-1}_{m+n}, \tag{3.9}$$

of functions L on a two-dimensional phase space (see [31]). As a distinguished subalgebra of (3.9) we have the set:

$$\mathrm{su}(1,1) = \{L_0 \equiv L^1_0 = \tfrac{1}{2}a\bar{a},\ L_+ \equiv L^1_1 = \tfrac{1}{2}a^2,\ L_- \equiv L^1_{-1} = \tfrac{1}{2}\bar{a}^2\}, \tag{3.10}$$

which provides an oscillator realization of the su(1, 1) Lie algebra generators $L_\pm, L_0$, in terms of a single bosonic variable, with commutation relations (3.16). With this notation, the functions L^I_m in (3.8) can also be written as:

$$L^I_{\pm|m|} = 2^{I-1}(L_0)^{I-|m|}(L_\pm)^{|m|}. \tag{3.11}$$

This expression will be generalized for arbitrary $U(N_+, N_-)$ groups in Eq. (4.1).

Following on the analysis of distinguished subalgebras of (3.9), we have the “wedge” subalgebra

$$w_\wedge \equiv \{L^I_m, I - |m| \geq 0\} \tag{3.12}$$

of polynomial functions of the $\mathrm{sl}(2,\mathbb{R})$ generators $L_0, L_\pm$, which can be formally extended beyond the wedge $I - |m| \geq 0$ by considering functions on the punctured complex plane with $I \geq 0$ and arbitrary m. To the last set belong the (conformal-spin-2) generators $L_n \equiv L^1_n$, $n \in \mathbb{Z}$, which close the Virasoro algebra without central extension,

$$\{L_m, L_n\} = \mathrm{i}(n - m)L_{m+n}, \tag{3.13}$$

and the (conformal-spin-1) generators $\phi_m \equiv L^0_m$, which close the non-extended Abelian Kac–Moody algebra,

$$\{\phi_m, \phi_n\} = 0. \tag{3.14}$$

In general, the higher-su(1, 1)-spin fields L_n^I have "conformal-spin" $s = I + 1$ and "conformal-dimension" n (the eigenvalue of L_0^1).

w-Algebras have been used as the underlying gauge symmetry of two-dimensional gravity models, and induced actions for these "w-gravities" have been written (see for example [3]). They turn out to be constrained Wess–Zumino–Witten models [32], as happens with standard induced gravity. The quantization procedure *deforms* the classical algebra w to the quantum algebra $\mathcal{W}$ due to the presence of anomalies —deformations of Moyal type of Poisson and symplectic-diffeomorphism algebras caused essentially by normal order ambiguities (see below). Also, generalizing the SL(2, $\mathbb{R}$) Kac–Moody hidden symmetry of Polyakov's induced gravity, there are SL(∞, $\mathbb{R}$) and GL(∞, $\mathbb{R}$) Kac–Moody hidden symmetries for $\mathcal{W}_\infty$ and $\mathcal{W}_{1+\infty}$ gravities, respectively [33]. Moreover, as already mentioned, the symmetry $\mathcal{W}_{1+\infty}$ appears to be useful in the classification of universality classes in the fractional quantum Hall effect.

The group-theoretic structure underlying these $\mathcal{W}$ algebras was elucidated in [10], where $\mathcal{W}_\infty$ and $\mathcal{W}_{1+\infty}$ appeared to be distinct members ($c = 0$ and $c = -1/4$ cases, respectively) of the one-parameter family $\mathcal{W}_\infty(c)$ of non-isomorphic [11,12] infinite-dimensional factor Lie-algebras of the SU(1, 1) tensor operators:

$$\begin{aligned}\hat{L}^I_{\pm|m|} &\propto \underbrace{[\hat{L}_\mp, [\hat{L}_\mp, \cdots, [\hat{L}_\mp,}_{I-|m|\text{ times}}(\hat{L}_\pm)^I]\cdots]] \\ &= (\mathrm{ad}_{\hat{L}_\mp})^{I-|m|}(\hat{L}_\pm)^I \sim \hat{L}_0^{I-|m|}\hat{L}_\pm^{|m|} + \mathrm{O}(\hbar),\end{aligned} \tag{3.15}$$

when extended beyond the wedge $I - m \geq 0$. The generators $\hat{L}_+ = \hat{X}_{12}$, $\hat{L}_- = \hat{X}_{21}$, $\hat{L}_0 = (\hat{X}_{22} - \hat{X}_{11})/2$, fulfil the standard su(1, 1) Lie-algebra commutation relations:

$$[\hat{L}_\pm, \hat{L}_0] = \pm\hbar\hat{L}_\pm, \qquad [\hat{L}_+, \hat{L}_-] = 2\hbar\hat{L}_0, \tag{3.16}$$

and $\hat{C} = (\hat{L}_0)^2 - \frac{1}{2}(\hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+)$ is the Casimir operator of su(1, 1). The structure constants for $\mathcal{W}_c$(su(1, 1)) can be written in terms of sl(2, $\mathbb{R}$) Clebsch–Gordan coefficients and generalized (Wigner) $6j$-symbols [10,13], and they have the general form:

$$[\hat{L}^I_m, \hat{L}^J_n]_c = \sum_{r=0}^{\infty} \hbar^{2r+1} f^{IJ}_{mn}(2r; c)\hat{L}^{I+J-(2r+1)}_{n+m} + \hbar^{2I} Q_I(n; c)\delta^{I,J}\delta_{n+m,0}\mathbb{I}, \tag{3.17}$$

where $\mathbb{I} \sim \hat{L}^0_0$ denotes a central generator and the central charges $Q_I(n; c)$ provide for the existence of central extensions. For example, $Q_1(n; c) = \frac{c}{12}(n^3 - n)$ reproduces the typical central extension in the Virasoro sector $I = 1$, and $Q_I(n; c)$ supplies central charges to all conformal-spins $s = I + 1$. Quantum deformations of the polynomial or "wedge" subalgebra (3.12) do not introduce true central extensions. The inclusion of central terms in (3.17) requires the formal extension of (3.12) beyond the wedge $I - |m| \geq 0$ (see [10]), that is, the consideration of non-polynomial functions (3.11) on the Cartan generator L_0.

Central charges provide the essential ingredient required to construct invariant geometric action functionals on coadjoint orbits of the corresponding groups. When applied to Virasoro and $\mathcal{W}$ algebras, they lead to Wess–Zumino–Witten models for *induced conformal gravities in* $1 + 1$ *dimensions* (see e.g. Ref. [32]). Also, local and non-local versions of the Toda systems emerge, as integrable dynamical systems, from a one-parameter family of ("quantum tori Lie") subalgebras of gl(∞) (see [34]). Infinite-dimensional analogues of rigid tops are discussed in [34] too; some of these systems give rise to "quantized" (magneto) hydrodynamic equations of an ideal fluid on a torus.

The leading order (O($\hbar$), $r=0$) structure constants $f^{IJ}_{mn}(0;c)=Jm-In$ in (3.17) reproduce the classical structure constants in (3.9). It is also precisely for the specific values of $c=0$ and $c=-\frac{1}{4}$ ($\mathcal{W}_\infty$ and $\mathcal{W}_{1+\infty}$, respectively) that the sequence of higher-order terms on the right-hand side of (3.17) turns out to be zero whenever $I+J-2r\leq 2$ and $I+J-2r\leq 1$, respectively. Therefore, $\mathcal{W}_\infty$ (resp. $\mathcal{W}_{1+\infty}$) can be consistently truncated to a closed algebra containing only those generators $\hat{L}^I_m$ with positive conformal-spins $s=I+1\geq 2$ (resp. $s=I+1\geq 1$).

The higher-order terms (O($\hbar^3$), $r\geq 1$) can be captured in a classical construction by extending the Poisson bracket (3.9) to the Moyal bracket

$$\{L^I_m, L^J_n\}_{\mathrm{M}} = L^I_m \star L^J_n - L^J_n \star L^I_m = \sum_{r=0}^{\infty} 2\frac{(\hbar/2)^{2r+1}}{(2r+1)!} P^{2r+1}(L^I_m, L^J_n), \tag{3.18}$$

where $L\star L' \equiv \exp\left(\frac{\hbar}{2}P\right)(L,L')$ is an *invariant associative $\star$-product* and

$$P^r(L,L') \equiv \Upsilon_{\imath_1 \jmath_1}\cdots\Upsilon_{\imath_r \jmath_r}\frac{\partial^r L}{\partial x_{\imath_1}\cdots\partial x_{\imath_r}}\frac{\partial^r L'}{\partial x_{\jmath_1}\cdots\partial x_{\jmath_r}}, \tag{3.19}$$

with $x\equiv(a,\bar{a})$ and $\Upsilon \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$. We set $P^0(L,L')\equiv L\cdot L'$, the ordinary (commutative) product of functions. Indeed, Moyal brackets where identified in [35] as the primary quantum deformation $\mathcal{W}_\infty$ of the classical algebra w_∞ of area-preserving diffeomorphisms of the cylinder. Also, the oscillator realization in (3.8) of the su(1, 1) Lie-algebra generators $L_\pm$, L_0 in terms of a single boson $(a,\bar{a})$ is related to the "symplecton" algebra $\mathcal{W}_\infty(-3/16)$ of Biedenharn and Louck [26] and the higher-spin algebra hs(2) of Vasiliev [15].

4. Extending the previous constructions to $U(N_+, N_-)$

4.1. Generalized w_∞ algebras

The generalization of previous constructions to arbitrary unitary groups proves to be quite unwieldy, and a canonical classification of $U(N)$-tensor operators has, so far, been proven to exist only for $U(2)$ and $U(3)$ (see [26] and references therein). Tensor labelling is provided in these cases by the Gel'fand–Weyl pattern for vectors in the carrier space of unitary irreducible representations of $U(N)$ (see later on Section 5.2).

In the letter [16], a set of $U(N_+, N_-)$-tensor operators was put forward and the Lie-algebra structure constants, for the particular case of the oscillator realization (2.4), were calculated through Moyal bracket (see later on Section 4.3). The chosen set of operators $\hat{L}^I_m$ in the universal enveloping algebra $\mathcal{U}(u(N_+, N_-))$ was a natural generalization of the su(1, 1)-tensor operators of Eq. (3.11), where now L_0 is be replaced by N Cartan generators $\hat{X}_{\alpha\alpha}$, $\alpha=1,\ldots,N$, and L_+, L_- are replaced by $N(N-1)/2$ "rising" generators $\hat{X}_{\alpha\beta}$, $\alpha<$

β and $N(N-1)/2$ "lowering" generators $\hat{X}_{\alpha\beta}, \alpha > \beta$, respectively. The explicit form of these operators is:

$$\hat{L}^{I}_{+|m|} \equiv \prod_{\alpha} (\hat{X}_{\alpha\alpha})^{I_\alpha - \left(\sum_{\beta>\alpha} |m_{\alpha\beta}| + \sum_{\beta<\alpha} |m_{\beta\alpha}|\right)/2} \prod_{\alpha<\beta} (\hat{X}_{\alpha\beta})^{|m_{\alpha\beta}|},$$
$$\hat{L}^{I}_{-|m|} \equiv \prod_{\alpha} (\hat{X}_{\alpha\alpha})^{I_\alpha - \left(\sum_{\beta>\alpha} |m_{\alpha\beta}| + \sum_{\beta<\alpha} |m_{\beta\alpha}|\right)/2} \prod_{\alpha<\beta} (\hat{X}_{\beta\alpha})^{|m_{\alpha\beta}|}. \tag{4.1}$$

The upper (generalized spin) index $I \equiv (I_1, \ldots, I_N)$ of $\hat{L}$ in (4.1) represents now a N-dimensional vector, which is taken to lie on a half-integral lattice $I_\alpha \in \mathbb{N}/2$; the lower index ("third component") m symbolizes now an integral upper-triangular $N \times N$ matrix,

$$m = \begin{pmatrix} 0 & m_{12} & m_{13} & \cdots & m_{1N} \\ 0 & 0 & m_{23} & \ldots & m_{2N} \\ 0 & 0 & 0 & \ldots & m_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \end{pmatrix}_{N\times N}, \quad m_{\alpha\beta} \in \mathbb{Z}, \tag{4.2}$$

and $|m|$ means absolute value of all its entries. Thus, the operators $\hat{L}^I_m$ are labelled by $N + N(N-1)/2 = N(N+1)/2$ indices, in the same way as wave functions ψ^I_m in the carrier space of unirreps of $U(N)$ (see Section 5.2). We shall not restrict ourselves to polynomial ("wedge") subalgebras

$$\mathcal{W}_\wedge(N_+, N_-) \equiv \left\{ \hat{L}^I_m, I_\alpha - \left(\sum_{\beta>\alpha} |m_{\alpha\beta}| + \sum_{\beta<\alpha} |m_{\beta\alpha}| \right) /2 \in \mathbb{N} \right\}, \tag{4.3}$$

and we shall consider "extensions beyond the wedge" (4.3) [to use the same nomenclature as the authors of Ref. [10] in the context of $\mathcal{W}$ algebras]; that is, we shall let the upper indices I_α take arbitrary half-integer values $I_\alpha \in \mathbb{N}/2$. This way, we are giving the possibility of true central extensions to the Lie algebra (4.4).[2]

The manifest expression of the structure constants f for the commutators

$$[\hat{L}^I_m, \hat{L}^J_n] = \hat{L}^I_m \hat{L}^J_n - \hat{L}^J_n \hat{L}^I_m = f^{IJl}_{mnK} \hat{L}^K_l \tag{4.4}$$

of a pair of operators (4.1) entails a cumbersome and awkward computation, because of inherent ordering problems. However, the essence of the full "quantum" algebra (4.4) can be still captured in a classical construction by extending the Poisson–Lie bracket (2.12) of a pair of functions L^I_m, L^J_n on the commuting coordinates $x_{\alpha\beta}$ to its deformed version, in the sense of Ref. [36]. To perform calculations with (2.12) is still rather complicated because of

[2] This claim deserves more careful study. So far, it is just an extrapolation of what happens to $\mathcal{W}_\infty$, Virasoro and Kac–Moody algebras, where Laurent (and not Taylor or polynomial) expansions provide couples of conjugated variables (positive and negative modes).

non-canonical brackets for the generating elements $x_{\alpha\beta}$. A way out to this technical problem is to make use of the classical analogue of the standard oscillator realization, $x_{\alpha\beta} = \bar{a}_\alpha a_\beta$, of the generators of $u(N_+, N_-)$, and replace the Poisson–Lie bracket (2.12) by the standard Poisson bracket

$$\{L^I_m, L^J_n\} = \mathrm{i}\Lambda_{\alpha\beta}\left(\frac{\partial L^I_m}{\partial a_\alpha}\frac{\partial L^J_n}{\partial \bar{a}_\beta} - \frac{\partial L^I_m}{\partial \bar{a}_\beta}\frac{\partial L^J_n}{\partial a_\alpha}\right), \tag{4.5}$$

for the Heisenberg–Weyl algebra $\{a_\alpha, \bar{a}_\beta\} = \mathrm{i}\Lambda_{\alpha,\beta}$. Although it is clear that, in general, both algebras are not isomorphic, we shall see that the difference entails just an ordering problem. Moreover, the bracket (4.5) has the advantage that simplifies calculations and expressions greatly. Indeed, it is not difficult to compute (4.5) which, after some algebraic manipulations, gives:

$$\{L^I_m, L^J_n\} = \mathrm{i}\Lambda^{\alpha\beta}(I_\alpha n_\beta - J_\alpha m_\beta)L^{I+J-\delta_\alpha}_{m+n}, \tag{4.6}$$

where

$$m_\alpha \equiv \left(\sum_{\beta>\alpha} m_{\alpha\beta} - \sum_{\beta<\alpha} m_{\beta\alpha}\right) \tag{4.7}$$

defines the components of a N-dimensional integral vector linked to the integral upper-triangular matrix m in (4.2), and

$$\delta_\alpha \equiv (\delta^1_\alpha, \ldots, \delta^N_\alpha) \tag{4.8}$$

is a N-dimensional vector with the αth entry equal to one and zero elsewhere. There is a clear resemblance between the w_∞ algebra (3.9) and (4.6), although the last one is far richer, as we shall show in Section 4.2. We shall refer to (4.6) as $w_\infty(N_+, N_-)$, or "generalized w_∞", algebra.

Let us see more carefully what we miss by replacing the Poisson–Lie bracket (2.12) with the standard Poisson bracket (4.5). First we note that the change of variable $x_{\alpha\beta} = \bar{a}_\alpha a_\beta$ in

$$\begin{aligned}
&\Lambda_{\alpha\beta}\left(\frac{\partial L}{\partial a_\alpha}\frac{\partial L'}{\partial \bar{a}_\beta} - \frac{\partial L}{\partial \bar{a}_\beta}\frac{\partial L'}{\partial a_\alpha}\right)\\
&\quad= \Lambda_{\alpha\beta}\left(\frac{\partial x_{\alpha_1\beta_1}}{\partial a_\alpha}\frac{\partial L}{\partial x_{\alpha_1\beta_1}}\frac{\partial x_{\alpha_2\beta_2}}{\partial \bar{a}_\beta}\frac{\partial L'}{\partial x_{\alpha_2\beta_2}} - \frac{\partial x_{\alpha_1\beta_1}}{\partial \bar{a}_\beta}\frac{\partial L}{\partial x_{\alpha_1\beta_1}}\frac{\partial x_{\alpha_2\beta_2}}{\partial a_\alpha}\frac{\partial L'}{\partial x_{\alpha_2\beta_2}}\right)\\
&\quad= \Lambda_{\alpha\beta}\left(\bar{a}_{\alpha_1}\delta^\alpha_{\beta_1}\frac{\partial L}{\partial x_{\alpha_1\beta_1}}a_{\beta_2}\delta^\beta_{\alpha_2}\frac{\partial L'}{\partial x_{\alpha_2\beta_2}} - a_{\beta_1}\delta^\beta_{\alpha_1}\frac{\partial L}{\partial x_{\alpha_1\beta_1}}\bar{a}_{\alpha_2}\delta^\alpha_{\beta_2}\frac{\partial L'}{\partial x_{\alpha_2\beta_2}}\right)\\
&\quad= (\Lambda_{\alpha_2\beta_1}x_{\alpha_1\beta_2} - \Lambda_{\alpha_1\beta_2}x_{\alpha_2\beta_1})\frac{\partial L}{\partial x_{\alpha_1\beta_1}}\frac{\partial L'}{\partial x_{\alpha_2\beta_2}},
\end{aligned} \tag{4.9}$$

is not one-to-one, as we have N^2 (real) coordinates $x_{\alpha\beta}$ and $2N$ (real) coordinates $a_\alpha, \bar{a}_\beta$. Also, the Poisson algebra (4.5) does not distinguish between polynomials like $x_{\alpha_1\beta_1}x_{\alpha_2\beta_2}$ and $x_{\alpha_1\beta_2}x_{\alpha_2\beta_1}$, which admit the same form when written in terms of the commuting oscillator variables $a_\alpha, \bar{a}_\beta$ as $x_{\alpha\beta} = \bar{a}_\alpha a_\beta$. That is, non-zero combinations like $x_{\alpha_1\beta_1}x_{\alpha_2\beta_2} - x_{\alpha_1\beta_2}x_{\alpha_2\beta_1}$

behave as zero under Poisson brackets (4.5). More precisely, we can see that "null-type" polynomials like:

$$x_{\alpha_1\beta_1\alpha_2\beta_2} \equiv x_{\alpha_1\beta_1}x_{\alpha_2\beta_2} - x_{\alpha_1\beta_2}x_{\alpha_2\beta_1} \tag{4.10}$$

generate ideals of the algebra $C^\infty(\mathcal{G}^*)$ of smooth functions L on the coalgebra $\mathcal{G}^*$. Indeed, it suffices to realize that the Poisson–Lie bracket between a generic monomial $x_{\alpha\beta}$ and a null-type polynomial (4.10) gives a combination of null-type polynomials, that is:

$$\begin{aligned}\{x_{\alpha\beta}, x_{\alpha_1\beta_1\alpha_2\beta_2}\}_{\mathrm{PL}} &= \mathrm{i}\Lambda_{\alpha_1\beta}x_{\alpha\beta_1\alpha_2\beta_2} - \mathrm{i}\Lambda_{\alpha\beta_1}x_{\alpha_1\beta\alpha_2\beta_2} + \mathrm{i}\Lambda_{\alpha_2\beta}x_{\alpha_1\beta_1\alpha\beta_2} \\ &\quad - \mathrm{i}\Lambda_{\alpha\beta_2}x_{\alpha_1\beta_1\alpha_2\beta},\end{aligned} \tag{4.11}$$

and similarly for general null-type polynomials of higher degree. Thus, we can say that the standard Poisson algebra (4.5) and (4.6) is a subalgebra of the quotient $C^\infty(\mathcal{G}^*)/\mathcal{I}$ of $C^\infty(\mathcal{G}^*)$ [with Poisson–Lie bracket (2.12)] by the ideal $\mathcal{I}$ generated by null-type polynomials.

This quotient captures the essence of the full algebra and will be enough for our purposes [we shall give in Section 5 the main guidelines to deal with the general case (general representations)]. Nevertheless, we should not forget that ordering problems like this are typically the origin of important central extensions and *anomalies* in quantum theory. Namely: Schwinger terms that appear in quantum current algebras, when currents are written in terms of fermionic matter operators; or central charges like $Q_1(n) = \frac{c}{12}(n^3 - n)$ for the Virasoro sector in (3.17), when the $\mathrm{diff}(S^1)$ generators $\hat{L}_n$ are written in terms of primary fields of WZW models, according to the Sugawara construction; or even the zero-point energy of the quantum harmonic oscillator, with important physical consequences like the Casimir effect, etc.

Before discussing quantum (Moyal) deformations of (4.6), let us recognize some of its relevant subalgebras.

4.2. *Distinguished subalgebras of $w_\infty(N_+, N_-)$*

There are many possible ways of embedding the $u(N_+, N_-)$ generators (2.6) inside (4.6), as there are also many possible choices of su(1, 1) inside (3.9). However, a "canonical" choice is:

$$\hat{X}_{\alpha\beta} \equiv -\mathrm{i}\hbar L^{\delta_\alpha}_{e_{\alpha\beta}}, \qquad e_{\alpha\beta} \equiv \mathrm{sign}(\beta - \alpha) \sum_{\sigma=\min(\alpha,\beta)}^{\max(\alpha,\beta)-1} e_{\sigma,\sigma+1}, \tag{4.12}$$

where δ_α is defined in (4.8) and $e_{\sigma,\sigma+1}$ denotes an upper-triangular matrix with the $(\sigma, \sigma+1)$-entry equal to one and zero elsewhere, that is $(e_{\sigma,\sigma+1})_{\mu\nu} = \delta_{\sigma,\mu}\delta_{\sigma+1,\nu}$ (we set $e_{\alpha\alpha} \equiv 0$). For example, the $u(1, 1)$ Lie-algebra generators correspond to:

$$\hat{X}_{12} = -\mathrm{i}\hbar L^{(1,0)}_{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \qquad \hat{X}_{21} = -\mathrm{i}\hbar L^{(0,1)}_{\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}},$$

$$\hat{X}_{11} = -\mathrm{i}\hbar L^{(1,0)}_{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}, \qquad \hat{X}_{22} = -\mathrm{i}\hbar L^{(0,1)}_{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}. \tag{4.13}$$

Letting the lower-index $m = e_{\alpha\beta}$ in (4.12) run over arbitrary integral upper-triangular matrices m, we arrive to the following infinite-dimensional algebra (as can be seen from (4.6)):

$$\{L_m^{\delta_\alpha}, L_n^{\delta_\beta}\} = -\mathrm{i}(m^\beta L_{m+n}^{\delta_\alpha} - n^\alpha L_{m+n}^{\delta_\beta}), \tag{4.14}$$

which we shall denote by $w_\infty^{(1)}(N_+, N_-)$. Reference [13] also considered infinite continuations of the particular finite-dimensional symmetries SO(1, 2) and SO(3, 2), as an "analytic continuation", i.e. an extension (or "revocation", to use their own expression) of the region of definition of the Lie-algebra generators' labels. It is easy to see that, for $u(1, 1)$, the "analytic continuation" (4.14) leads to two Virasoro sectors: $L_{m_{12}} \equiv L_m^{(1,0)}$, $\bar{L}_{m_{12}} \equiv L_m^{(0,1)}$. Its $(3+1)$-dimensional counterpart $w_\infty^{(1)}(2, 2)$ contains four non-commuting Virasoro-like sectors $w_\infty^{(1_\alpha)}(2, 2) = \{L_m^{\delta_\alpha}\}$, $\alpha = 1, \ldots, 4$ which, in their turn, hold three genuine Virasoro sectors for $m = ku_{\alpha\beta}$, $k \in \mathbb{Z}$, $\alpha < \beta = 2, \ldots, 4$, where $u_{\alpha\beta}$ denotes an upper-triangular matrix with components $(u_{\alpha\beta})_{\mu\nu} = \delta_{\alpha,\mu}\delta_{\beta,\nu}$. In general, $w_\infty^{(1)}(N_+, N_-)$ contains $N(N-1)$ distinct and non-commuting Virasoro sectors,

$$\{V_k^{(\alpha\beta)}, V_l^{(\alpha\beta)}\} = -\mathrm{i}\Lambda^{\alpha\alpha}\mathrm{sign}(\beta - \alpha)(k - l)V_{k+l}^{(\alpha\beta)}, \quad V_k^{(\alpha\beta)} \equiv L_{ku_{\alpha\beta}}^{\delta_\alpha}, \tag{4.15}$$

and holds $u(N_+, N_-)$ as the *maximal finite-dimensional subalgebra*.

The algebra $w_\infty^{(1)}(N_+, N_-)$ can be seen as the *minimal* infinite continuation of $u(N_+, N_-)$ representing the diffeomorphism algebra diff(N) of the N-torus $U(1)^N$. Indeed, the algebra (4.14) formally coincides with the algebra of vector fields $L_{f(y)}^\mu = f(y)\frac{\partial}{\partial y_\mu}$, where $y = (y_1, \ldots, y_N)$ denotes a local system of coordinates and $f(y)$ can be expanded in a plane wave basis, such that $L_{\vec{m}}^\mu = \mathrm{e}^{\mathrm{i}m^\alpha y_\alpha}\frac{\partial}{\partial y_\mu}$ constitutes a basis of vector fields for the so-called generalized Witt algebra [37],

$$[L_{\vec{m}}^\alpha, L_{\vec{n}}^\beta] = -\mathrm{i}(m^\beta L_{\vec{m}+\vec{n}}^\alpha - n^\alpha L_{\vec{m}+\vec{n}}^\beta), \tag{4.16}$$

of which there are studies about its representations (see e.g. [38]). Note that, for us, the N-dimensional lattice vector $\vec{m} = (m_1, \ldots, m_N)$ in (4.7) is, by construction, constrained to $\sum_{\alpha=1}^N m_\alpha = 0$ (i.e. $L_{\vec{m}}^\mu$ is *divergence free*), which introduces some novelties in (4.14) as regards the Witt algebra (4.16). Actually, the algebra (4.14) can be split into one "temporal" piece, constituted by an Abelian ideal generated by $\tilde{L}_m^N \equiv \Lambda_{\alpha\alpha}L_m^{\delta_\alpha}$, and a "residual" symmetry generated by the spatial diffeomorphisms

$$\tilde{L}_m^j \equiv \Lambda_{jj}L_m^{\delta_j} - \Lambda_{j+1,j+1}L_m^{\delta_{j+1}}, \quad j = 1, \ldots, N-1 \text{ (no sum on } j), \tag{4.17}$$

which act semi-directly on the temporal part. More precisely, the commutation relations (4.14) in this new basis adopt the following form:

$$\begin{aligned} &\{\tilde{L}_m^j, \tilde{L}_n^k\} = -\mathrm{i}(\tilde{m}^k\tilde{L}_{m+n}^j - \tilde{n}^j\tilde{L}_{m+n}^k), \qquad \{\tilde{L}_m^j, \tilde{L}_n^N\} = \mathrm{i}\tilde{n}^j\tilde{L}_{m+n}^N, \\ &\{\tilde{L}_m^N, \tilde{L}_n^N\} = 0, \end{aligned} \tag{4.18}$$

where $\tilde{m}_k \equiv m_k - m_{k+1}$. Only for $N = 2$, the last commutator admits a central extension of the form $\sim n_{12}\delta_{m+n,0}$ compatible with the rest of commutation relations (4.18). This result amounts to the fact that the (unconstrained) diffeomorphism algebra $\mathrm{diff}(N)$ does not admit any non-trivial central extension except when $N = 1$ (see [39]).

Another important point is in order here. The expression (4.12) reveals an embedding of the Lie algebra $u(N_+, N_-)$ inside the diffeomorphism algebra $\mathrm{diff}(N_+, N_-)$ with commutation relations (4.14). That is, this new way of labelling $u(N_+, N_-)$ generators provides an straightforward "analytic continuation" from $u(N_+, N_-)$ to $\mathrm{diff}(N_+, N_-)$.

As well as the "$U(N_+, N_-)$-spin $I = \delta_\mu$ currents" (diffeomorphisms) $L_m^{\delta_\mu}$ in (4.14), one can also introduce "higher-$U(N_+, N_-)$-spin I currents" L_m^I (in a sense similar to that of Ref. [14]) by letting the upper-index I run over an arbitrary half-integral N-dimensional lattice. Diffeomorphisms $L_m^{\delta_\mu}$ act semi-directly on "$u(N_+, N_-)$-spin J currents" L_n^J as follows (see Eq. (4.6)):

$$\{L_m^{\delta_\mu}, L_n^J\} = -\mathrm{i}\Lambda^{\alpha\beta} J_\alpha m_\beta L_{m+n}^{J+\delta_\mu-\delta_\alpha} + \mathrm{i}n^\mu L_{m+n}^J. \tag{4.19}$$

Note that this action leaves stable Casimir quantum numbers like the trace $\sum_{\alpha=1}^N J_\alpha$ [Casimir C_1 eigenvalue (2.7)]. This higher-spin structure of the algebra $w_\infty(N_+, N_-)$ will be justified and highlighted in Section 5, where higher-spin representations of pseudo-unitary groups will be explicitly calculated.

4.3. Quantum (Moyal) deformations

As it happens with w_∞-algebras, the quantization procedure, which entails unavoidable renormalizations (mainly due to ordering problems), must deform the classical ($\hbar \to 0$) "generalized w_∞" algebra $w_\infty(N_+, N_-)$ in (4.6) to a quantum algebra $\mathcal{W}_\infty(N_+, N_-)$, by adding higher-order (Moyal-type) terms and central extensions like in (3.17). There is basically only one possible deformation $\mathcal{W}_\infty(N_+, N_-)$ of the bracket (4.5) – corresponding to a full symmetrization – that fulfils the Jacobi identities (see Ref. [36]), which is the Moyal bracket (3.18,3.19), where now

$$\Upsilon \equiv \begin{pmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{pmatrix}$$

is a $2N \times 2N$ symplectic matrix. The calculation of higher-order terms in (3.18) is an arduous task, but the result can be summed up as follows:

$$\{L_m^I, L_n^J\}_{\mathrm{M}} = \sum_{r=0}^{\infty} 2\left(\frac{\hbar}{2}\right)^{2r+1} f_{\alpha_1\cdots\alpha_{2r+1}}^{\alpha_1\cdots\alpha_{2r+1}}(I, m; J, n) L_{m+n}^{I+J-\sum_{j=1}^{2r+1}\delta_{\alpha_j}}, \tag{4.20}$$

where the higher-order structure constants

$$f_{\alpha_1\cdots\alpha_{2r+1}}^{\alpha_1\cdots\alpha_{2r+1}}(I, m; J, n) \equiv \sum_{\ell=0}^{2r+1} \frac{(-1)^\ell}{(2r+1-\ell)!\ell!} \prod_{s=1}^{2r+1} \Lambda^{\alpha_s\beta_s} \Gamma_{\alpha_s}^\ell(I, -m)\Gamma_{\beta_s}^\ell(J, n) \tag{4.21}$$

are expressed in terms of the factors

$$\Gamma^{\ell}_{\alpha_s}(I, m) \equiv I^{(s)}_{\alpha_s} + (-1)^{\theta(\ell-s)} m_{\alpha_s}/2, \tag{4.22}$$

which are defined through the vectors (4.7) and $U(N_+, N_-)$-spins

$$I^{(s)}_{\alpha_s} = I_{\alpha_s} - \sum_{t=\theta(s-\ell-1)\ell+1}^{s-1} \delta^{\alpha_t}_{\alpha_s}, \qquad I^{(0)} = I^{(\ell+1)} \equiv I, \tag{4.23}$$

with

$$\theta(\ell - s) = \begin{cases} 0 \text{ if } \ell < s, \\ 1 \text{ if } \ell \geq s, \end{cases} \tag{4.24}$$

the Heaviside function. For example, for $r = 0$, the leading order (classical, $\hbar \to 0$) structure constants are:

$$\begin{aligned} f^{\alpha}_{\alpha}(I, m; J, n) &= \Lambda^{\alpha\beta}(\Gamma^0_{\alpha}(I, -m)\Gamma^0_{\beta}(J, n) - \Gamma^1_{\alpha}(I, -m)\Gamma^1_{\beta}(J, n)) \\ &= \Lambda^{\alpha\beta}((I_\alpha - m_\alpha/2)(J_\beta + n_\beta/2) - (I_\alpha + m_\alpha/2)(J_\beta - n_\beta/2)), \end{aligned} \tag{4.25}$$

which, after simplification, coincides with (4.6).

We have rephrased our previous (hard) problem of computing the commutators (4.4) of the tensor operators (4.1) in terms of (more easy) Moyal brackets of functions on the coalgebra $u(N_+, N_-)^*$ [up to quotients by the ideals $\mathcal{I}$ generated by "null-type" polynomials like (4.10)]. Nevertheless, Moyal bracket captures the essence of more general deformations, which may include central extensions like

$$\begin{aligned} [\hat{L}^I_m, \hat{L}^J_n] &= \hbar\Lambda^{\alpha\beta}(J_\alpha m_\beta - I_\alpha n_\beta)\hat{L}^{I+J-\delta_\alpha}_{m+n} + \mathrm{O}(\hbar^3) \\ &\quad + \hbar^{(\sum_{\alpha=1}^{N} I_\alpha + J_\alpha)} Q_I(m)\delta^{I,J}\delta_{m+n,0}\mathbb{I}, \end{aligned} \tag{4.26}$$

with central charges $Q_I(m)$ for all $U(N_+, N_-)$-spin I currents $\hat{L}^I_m$. Note that, the structure of this central extension implies that the modes $\hat{L}^I_m$ and $\hat{L}^I_{-m}$ are conjugated, a fact inherited from the conjugation relation $\hat{X}^\dagger_{\alpha\beta} = \hat{X}_{\beta\alpha}$ after (2.5) and the definition (4.1) of $\hat{L}^I_m$. An exhaustive study of this central extensions is in progress. Note that the diffeomorphism subalgebra $w^{(1)}_\infty(N_+, N_-)$ remains unaltered by Moyal deformations.

5. Towards a geometrical interpretation of $\mathcal{W}_\infty(N_+, N_-)$

In this section we want to highlight the higher-spin structure of $\mathcal{W}_\infty(N_+, N_-)$. To justify this view, we shall develop the representation theory of $U(N_+, N_-)$ (discrete series), calculating higher-spin representations, coherent states and deriving Kähler structures on flag manifolds, which are essential ingredients to define operator symbols.

5.1. Complex coordinates on flag manifolds

Although we shall restrict ourselves to the compact SU(N) case in the following general discussion, most of the results are easily extrapolated to the non-compact SU(N_+, N_-) case. Actually, we shall exemplify our construction with the $(3+1)$-dimensional conformal group $SU(2,2) = SO(4,2)$.

In order to put coordinates on $G = SU(N)$, the ideal choice is the Bruhat decomposition [40] for the coset space (flag manifold) $\mathbb{F} = G/T$, where we denote $T = U(1)^{N-1}$ the maximal torus. We shall introduce a local complex parametrization of $\mathbb{F}$ by means of the isomorphism $G/T = G^{\mathbb{C}}/B$, where $G^{\mathbb{C}} \equiv SL(N, \mathbb{C})$ is the complexification of G, and B is the Borel subgroup of upper triangular matrices. In one direction, the element $[g]_T \in G/T$ is mapped to $[g]_B \in G^{\mathbb{C}}/B$. For example, for $G = SU(4)$ we have:

$$[g]_T = \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \\ u_{31} & u_{32} & u_{33} & u_{34} \\ u_{41} & u_{42} & u_{43} & u_{44} \end{pmatrix} \end{array} \longrightarrow [g]_B = \begin{array}{c} \begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ z_{21} & 1 & 0 & 0 \\ z_{31} & z_{32} & 1 & 0 \\ z_{41} & z_{42} & z_{43} & 1 \end{pmatrix} \end{array}, \quad (5.1)$$

where

$$\begin{aligned} z_{21} &= \frac{u_{21}}{u_{11}}, \qquad z_{31} = \frac{u_{31}}{u_{11}}, \qquad z_{41} = \frac{u_{41}}{u_{11}}, \\ z_{32} &= \frac{u_{11}u_{32} - u_{12}u_{31}}{u_{11}u_{22} - u_{12}u_{21}}, \qquad z_{42} = \frac{u_{11}u_{42} - u_{12}u_{41}}{u_{11}u_{22} - u_{12}u_{21}}, \\ z_{43} &= \frac{u_{13}(u_{21}u_{42} - u_{22}u_{41}) - u_{23}(u_{11}u_{42} - u_{12}u_{41}) + u_{43}(u_{11}u_{22} - u_{12}u_{21})}{u_{13}(u_{21}u_{32} - u_{22}u_{31}) - u_{23}(u_{11}u_{32} - u_{12}u_{31}) + u_{33}(u_{11}u_{22} - u_{12}u_{21})}, \end{aligned} \quad (5.2)$$

provides a complex coordinatization $\{z_{\alpha\beta}, \alpha > \beta = 1, 2, 3\}$ of nearly all of the six-dimensional flag manifold $\mathbb{F}_3 = SU(4)/U(1)^3$, missing only a lower-dimensional subspace; indeed, these coordinates are defined where the denominators are non-zero. In general, each flag $\mathbb{F}_{N-1}$ is covered by $N!$ patches, related to the elements of the Weyl group of G: the symmetric group S_N of N elements. A complete atlas of coordinate charts is obtained by moving this coordinate patch around by means of left multiplication with the Weyl group representatives (see e.g. [42]). We shall restrict ourselves to the largest Bruhat cell (5.1).

In the other direction, i.e. from $G^{\mathbb{C}}/B$ to G/T, one uses the Iwasawa decomposition: any element $g^c \in G^{\mathbb{C}}$ may be factorized as $g^c = gb$, $g \in G$, $b \in B$ in a unique fashion, up to torus elements $t \in T$ (the Cartan subgroup of diagonal matrices $t = \mathrm{diag}(t_1, t_2/t_1, t_3/t_2, \ldots, 1/t_{N-1})$), which coordinates t_α can be calculated as the arguments $t_\alpha = (\Delta_\alpha(g)/\bar{\Delta}_\alpha(g))^{1/2}$ of the α-upper principal minors Δ_α of $g \in G$. For example, for SU(4) we have:

$$t_1 = \left(\frac{u_{11}}{\bar{u}_{11}}\right)^{1/2}, \qquad t_2 = \left(\frac{u_{11}u_{22} - u_{12}u_{21}}{\bar{u}_{11}\bar{u}_{22} - \bar{u}_{12}\bar{u}_{21}}\right)^{1/2},$$

$$t_3 = \left(\frac{u_{13}(u_{21}u_{32} - u_{22}u_{31}) - u_{23}(u_{11}u_{32} - u_{12}u_{31}) + u_{33}(u_{11}u_{22} - u_{12}u_{21})}{\bar{u}_{13}(\bar{u}_{21}\bar{u}_{32} - \bar{u}_{22}\bar{u}_{31}) - \bar{u}_{23}(\bar{u}_{11}\bar{u}_{32} - \bar{u}_{12}\bar{u}_{31}) + \bar{u}_{33}(\bar{u}_{11}\bar{u}_{22} - \bar{u}_{12}\bar{u}_{21})} \right)^{1/2}. \tag{5.3}$$

The Iwasawa decomposition in this case may be proved by means of the Gram–Schmidt ortonormalization process: regard any $g^c = [g]_B \in G^{\mathbb{C}}$ [like the one in (5.1)] as a juxtaposition of N column vectors $(z_1, z_2, \ldots, z_N)$. Then one obtains orthogonal vectors $\{v_\alpha\}$ in the usual way:

$$v'_\alpha = \left(z_\alpha - \frac{(z_\alpha, v_{\alpha-1})}{(v'_{\alpha-1}, v'_{\alpha-1})} v'_{\alpha-1} - \cdots - \frac{(z_\alpha, v_1)}{(v'_1, v'_1)} v'_1 \right), \qquad v_\alpha = \frac{v'_\alpha}{(\Lambda^{\alpha\alpha}(v'_\alpha, v'_\alpha))^{1/2}} \tag{5.4}$$

(not sum on α) where $(z_\alpha, v_\beta) \equiv \bar{z}_{\alpha\mu} \Lambda^{\mu\nu} v_{\beta\nu}$ denotes a scalar product with metric Λ. At this point, it should be noted that the previous procedure can be straightforwardly extended to the non-compact case $G = \mathrm{SU}(N_+, N_-)$ just by considering the indefinite metric $\Lambda = \mathrm{diag}(1, \ldots^{N_+}, 1, -1, \ldots^{N_-}, -1)$. Using a relativistic notation, we may say that the vectors $v_1, \ldots, v_{N_+}$ are "space-like" [that is, $(v_\alpha, v_\beta) = 1$] whereas $v_{N_+ +1}, \ldots, v_N$ are "time-like" [i.e, $(v_\alpha, v_\beta) = -1$]; this ensures that $v \Lambda v^\dagger = \Lambda$. For example, for SU(2, 2), the explicit expression of (5.4) proves to be:

$$\begin{aligned}
v_1 &= |\Delta_1| \begin{pmatrix} 1 \\ z_{21} \\ z_{31} \\ z_{41} \end{pmatrix}, \\
v_2 &= |\Delta_1||\Delta_2| \begin{pmatrix} -\bar{z}_{21} + z_{32}\bar{z}_{31} + z_{42}\bar{z}_{41} \\ 1 + z_{32}z_{21}\bar{z}_{31} - z_{31}\bar{z}_{31} + z_{42}z_{21}\bar{z}_{41} - z_{41}\bar{z}_{41} \\ z_{32} + z_{32}z_{21}\bar{z}_{21} - \bar{z}_{21}z_{31} + z_{42}z_{31}\bar{z}_{41} - z_{32}z_{41}\bar{z}_{41} \\ z_{42} + z_{42}z_{21}\bar{z}_{21} - z_{42}z_{31}\bar{z}_{31} - \bar{z}_{21}z_{41} + z_{32}\bar{z}_{31}z_{41} \end{pmatrix}, \\
v_3 &= |\Delta_2||\Delta_3| \begin{pmatrix} [-\bar{z}_{32}\bar{z}_{21} - \bar{z}_{42}z_{43}\bar{z}_{21} + \bar{z}_{31} - z_{42}\bar{z}_{42}\bar{z}_{31} + z_{32}\bar{z}_{42}z_{43}\bar{z}_{31} \\ +\bar{z}_{32}z_{42}\bar{z}_{41} + z_{43}\bar{z}_{41} - z_{32}\bar{z}_{32}z_{43}\bar{z}_{41}] \\ [\bar{z}_{32} + \bar{z}_{42}z_{43} - z_{42}\bar{z}_{42}z_{21}\bar{z}_{31} + z_{32}\bar{z}_{42}z_{43}z_{21}\bar{z}_{31} - \bar{z}_{42}z_{43}z_{31}\bar{z}_{31} + \bar{z}_{42}\bar{z}_{31}z_{41} \\ +\bar{z}_{32}z_{42}z_{21}\bar{z}_{41} - z_{32}\bar{z}_{32}z_{43}z_{21}\bar{z}_{41} + \bar{z}_{32}z_{43}z_{31}\bar{z}_{41} - \bar{z}_{32}z_{41}\bar{z}_{41}] \\ [1 - z_{42}\bar{z}_{42} + z_{32}\bar{z}_{42}z_{43} - z_{42}\bar{z}_{42}z_{21}\bar{z}_{21} + z_{32}\bar{z}_{42}z_{43}z_{21}\bar{z}_{21} - \bar{z}_{42}z_{43}\bar{z}_{21}z_{31} \\ +\bar{z}_{42}\bar{z}_{21}z_{41} + z_{42}z_{21}\bar{z}_{41} - z_{32}z_{43}z_{21}\bar{z}_{41} + z_{43}z_{31}\bar{z}_{41} - z_{41}\bar{z}_{41}] \\ [\bar{z}_{32}z_{42} + z_{43} - z_{32}\bar{z}_{32}z_{43} + \bar{z}_{32}z_{42}z_{21}\bar{z}_{21} - z_{32}\bar{z}_{32}z_{43}z_{21}\bar{z}_{21} + \bar{z}_{32}z_{43}\bar{z}_{21}z_{31} \\ -z_{42}z_{21}\bar{z}_{31} + z_{32}z_{43}z_{21}\bar{z}_{31} - z_{43}z_{31}\bar{z}_{31} - \bar{z}_{32}\bar{z}_{21}z_{41} + \bar{z}_{31}z_{41}] \end{pmatrix}, \\
v_4 &= |\Delta_3| \begin{pmatrix} -\bar{z}_{42}\bar{z}_{21} + \bar{z}_{32}\bar{z}_{43}\bar{z}_{21} - \bar{z}_{43}\bar{z}_{31} + \bar{z}_{41} \\ \bar{z}_{42} - \bar{z}_{32}\bar{z}_{43} \\ -\bar{z}_{43} \\ 1 \end{pmatrix},
\end{aligned} \tag{5.5}$$

where

$$|\Delta_1(z,\bar{z})| = \frac{1}{\sqrt{1+|z_{21}|^2-|z_{31}|^2-|z_{41}|^2}},$$
$$|\Delta_2(z,\bar{z})| = \frac{1}{\sqrt{1+|z_{32}z_{41}-z_{42}z_{31}|^2-|z_{32}|^2-|z_{42}|^2-|z_{32}z_{21}-z_{31}|^2-|z_{42}z_{21}-z_{41}|^2}},$$
$$|\Delta_3(z,\bar{z})| = \frac{1}{\sqrt{1+|z_{43}|^2-|z_{42}-z_{43}z_{32}|^2-|z_{41}+z_{43}z_{32}z_{21}-z_{42}z_{21}-z_{43}z_{31}|^2}} \tag{5.6}$$

are the moduli of the $\alpha = 1, 2, 3$ upper principal minors $\Delta_\alpha(g)$ of $g \in G$. These "characteristic lengths" will play a central role in what follows.

Any (peudo-) unitary matrix $g \in G$ in the present patch (which contains the identity element $z = 0 = \bar{z}, t = 1$) can be written in minimal coordinates $g = (z_{\alpha\beta}, \bar{z}_{\alpha\beta}, t_\beta), \alpha > \beta = 1, \ldots, N-1$, as the product $g = vt$ of an element v of the base (flag) $\mathbb{F}$ times an element t of the fibre $T = U(1)^{N-1}$.

Once we have the expression of a general G group element $g = (g^1, \ldots, g^{N^2-1})$ in terms of the minimal coordinates $g = (z_{\alpha\beta}, \bar{z}_{\alpha\beta}, t_\beta), \alpha > \beta = 1, \ldots, N-1$, we can easily write the group law $g'' = g' \bullet g$ and compute the left- and right-invariant vector fields

$$\left.\begin{aligned} X^L_j(g) &\equiv \mathcal{L}^k_j(g)\frac{\partial}{\partial g^k}, \qquad \mathcal{L}^k_j(g) = \left.\frac{\partial(g\bullet g')^k}{\partial g'^j}\right|_{g'=e} \\ X^R_j(g) &\equiv \mathcal{R}^k_j(g)\frac{\partial}{\partial g^k}, \qquad \mathcal{R}^k_j(g) = \left.\frac{\partial(g'\bullet g)^k}{\partial g'^j}\right|_{g'=e} \end{aligned}\right\}$$
$$j, k = 1, \ldots, N^2-1 = \dim(G). \tag{5.7}$$

The algebraic correspondence between right-invariant vector fields and the step operators (2.2), with commutation relations (2.5), is:

$$X^R_{z_{\alpha\beta}} \to \hat{X}_{\alpha\beta}, \qquad X^R_{\bar{z}_{\alpha\beta}} \to \hat{X}_{\beta\alpha}, \qquad X^R_{t_\beta} \to \hat{X}_{\beta\beta} - \hat{X}_{\beta+1,\beta+1},$$
$$\alpha > \beta = 1, \ldots, N-1. \tag{5.8}$$

5.2. *Higher-spin representations, coherent states and Kähler structures on flag manifolds*

In this section we shall compute the unitary irreducible representations of G and we shall construct coherent states and geometric structures attached to them. Let us start by considering the (finite) left regular representation $[L_g\Psi](g') = \Psi(g^{-1}\bullet g')$ of the group G on complex functions Ψ on G [remember Eq. (2.8)]. This representation is highly reducible. The reduction can be achieved through a complete set of finite right restrictions or "polarization equations" (in the language of geometric quantization [43]):

$$[R_g\Psi](g') = \Psi(g'\bullet g) = D^c(g)\Psi(g') \quad \forall g \in P\ \forall g' \in G, \tag{5.9}$$

which impose that Ψ must transform according to a given (Abelian) representation D^c (with index c) of a certain maximal proper subgroup $P \subset G$ ("polarization subgroup").

The Lie algebra $\mathcal{P}$ of P is called a "first-order polarization", which formal definition could be stated as in the following definition.

Definition 5.1. A first-order polarization is a proper subalgebra $\mathcal{P}$ of the Lie algebra $\mathcal{G}$ of G, realized in terms of left-invariant vector fields X^L [the infinitesimal generators of finite right translations (5.9)]. It must satisfy a maximality condition in order to define an irreducible representation of G.

Hence, at the Lie algebra level, the polarization equations (5.9) acquire the form of a system of non-homogeneous first-order partial differential equations:[3]

$$X^L_j \Psi = c_j \Psi(g) \quad \forall X^L_j \in \mathcal{P}, \tag{5.10}$$

where c denotes a one-dimensional representation (character) of the polarization subalgebra $\mathcal{P}$, $c(X^L_i) = c_i \, \forall X^L_i \in \mathcal{P}$. That is, c is the infinitesimal character associated to D^c in (5.9). Notice that since the representation is one-dimensional, the character c vanishes on the derived subalgebra $[\mathcal{P}, \mathcal{P}]$ of $\mathcal{P}$, i.e. $c([X^L_i, X^L_j]) = 0 \, \forall X^L_i, X^L_j \in \mathcal{P}$. This means that it factorizes via the Abelian quotient $\mathcal{P}/[\mathcal{P}, \mathcal{P}]$. Hence the value of c on $\mathcal{P}$ is determined by the value of the factorized $\bar{c}$ on the Abelian quotient. For our case, the first-order polarization subalgebra $\mathcal{P}$ will be generated by the following $(N-1+N(N-1)/2)$ left-invariant vector fields:

$$\mathcal{P} = \langle X^L_{t_\beta}, X^L_{z_{\alpha\beta}}, \alpha > \beta = 1, \ldots, N-1 \rangle.$$

Then, the quotient $\mathcal{P}/[\mathcal{P}, \mathcal{P}]$ coincides here with the Abelian Cartan subalgebra $\mathcal{T} = u(1)^{N-1}$. Therefore, denoting by $c(X^L_{t_\beta}) \equiv -2S_\beta \, \forall X^L_{t_\beta} \in \mathcal{T}$ the non-zero characters or "G-spin labels", the solution to the polarization equations (5.10),

$$\left.\begin{array}{l} X^L_{t_\beta} \Psi = -2S_\beta \Psi \\ X^L_{z_{\alpha\beta}} \Psi = 0, \end{array}\right\} \Psi^S(g) = W_S(g) \Phi(\bar{z}), \tag{5.11}$$

can be arranged as the product of a highest-weight vector W_S ("vacuum"), which is a particular solution of $X^L_{t_\beta} \Psi = -2S_\beta \Psi$ and can be written as a product of upper principal minors

$$W_S(g) \equiv \prod_{\beta=1}^{N-1} (\bar{\Delta}_\beta(g))^{2S_\beta}, \tag{5.12}$$

times an anti-holomorphic function $\Phi(\bar{z})$, which can be written as an analytic power series, with complex coefficients a^S_m, on its arguments $\bar{z}_{\alpha\beta}$,

$$\Phi(\bar{z}) \equiv \sum_m a^S_m \prod_{\alpha>\beta} (\bar{z}_{\alpha\beta})^{m_{\beta\alpha}}. \tag{5.13}$$

[3] This procedure for obtaining irreducible representations resembles Mackey's induction method, except for the fact that it can be extended to "higher-order polarizations": subalgebras $\mathcal{P}^{HO}$ of the (left) universal enveloping algebra $\mathcal{U}(G)$ which also satisfy a maximality condition in order to define an irreducible representation (see e.g. [41] for more details).

The index m denotes an integral upper-triangular $N \times N$ matrix [see (4.2)]. The range of the entries $m_{\alpha\beta}, \alpha < \beta = 2, \ldots, N$ depends on the set of G-spin indices $\{S_\beta\}_{\beta=1}^{N-1}$, which label particular G-spin S irreducible representations of G on the Hilbert space $\mathcal{H}_S(G)$ of polarized wave functions (5.11).

The sign of the SU(N_+, N_-)-spin indices S_β depends on the (non-)compact character of the corresponding simple roots: the ones whose generators $X_{\alpha\beta}$ fulfil $\beta = \alpha + 1$. With this notation, all the roots $(\alpha\beta)$ are of compact type except for $(\alpha\beta) = (N_+, N_+ + 1)$. This fact implies that $S_\beta \in \mathbb{Z}^+/2$ except for $S_{N_+} \in \mathbb{Z}^-/2$. Indeed, with this choice of sign we guarantee: (a) the *finiteness* of the scalar product $\langle \Phi|\Psi\rangle \equiv \int_G \mathrm{d}^L g\, \bar{\Phi}(g)\Psi(g)$, which Haar measure has the form:

$$\mathrm{d}^L g = \prod_{\beta=1}^{N-1} |\Delta_\beta(z, \bar{z})|^4 \bigwedge_{\beta=1}^{N-1} t_\beta^{-1}\, \mathrm{d}t_\beta \bigwedge_{\alpha>\beta} \mathrm{d}z_{\alpha\beta} \wedge \mathrm{d}\bar{z}_{\alpha\beta} \tag{5.14}$$

[where we have used that $\det(\mathcal{L}_j^k(g))^{-1} = \prod_{\beta=1}^{N-1} |\Delta_\beta(z, \bar{z})|^4 t_\beta^{-1}$] and (b) the *unitarity* of the representation $[L_{g'}\Psi](g) = \Psi(g'^{-1} \bullet g)$ of G. We can still keep track of the extra $U(1)$ quantum number S_N that differentiates $U(N) \simeq (\mathrm{SU}(N) \times U(1))/\mathbb{Z}_N$ from SU(N) representations. The $U(N)$ wave functions $\tilde{\Psi}^I$ depend on an extra $U(1)$-factor $(t_N)^{-2S_N}, t_N \in U(1)$ in the vacuum wave function W_S in (5.13), where the relation between the $U(N)$-spin labels $I = (I_1, \ldots, I_N)$ of Eq. (4.1) and the SU$(N) \times U(1)$-spin labels $S = (S_1, \ldots, S_N)$ is: $S_\beta = I_\beta - I_{\beta+1}, \beta = 1, \ldots, N-1$ and $S_N = \sum_{\alpha=1}^{N} I_\alpha$ [the Casimir C_1 (trace) eigenvalue].

The basic wave functions $\Psi_m^S(g) \equiv W_S(g) \prod_{\alpha>\beta} (\bar{z}_{\alpha\beta})^{m_{\beta\alpha}}$ of $\mathcal{H}_S(G)$ are eigenfunctions of the right-invariant differential operators $X_{t_\beta}^R$ (Cartan generators):

$$X_{t_\beta}^R \Psi_m^S = (2S_\beta + m_\beta - m_{\beta+1})\Psi_m^S, \tag{5.15}$$

where m_β is defined in (4.7); notice that the eigenvalue $(2S_\beta + m_\beta - m_{\beta+1})$ of $X_{t_\beta}^R$ can also be written as $2(\Gamma_\beta^0(S, m) - \Gamma_{\beta+1}^0(S, m))$, where $\Gamma_\beta^0(S, m)$ is one of the characteristic factors (4.22) that appears in the power expansion of the structure constants (4.21) of the algebra (4.20). The lowering operators $Z_{\alpha\beta} \equiv X_{z_{\alpha\beta}}^R$ annihilate the vacuum vector $\Psi_0^S = W_S$. The rest of vectors $L_m^S(g)$ of the Hilbert space $\mathcal{H}_S(G)$ can be obtained through the orbit of the vacuum under the action of rising operators $Z_{\alpha\beta}^\dagger \equiv X_{\bar{z}_{\alpha\beta}}^R$:

$$L_m^S(g) \equiv \prod_{\alpha>\beta} (Z_{\alpha\beta}^\dagger)^{m_{\beta\alpha}} W_S(g), \quad m_{\alpha\beta} \in \mathbb{N}. \tag{5.16}$$

Notice that the way of labelling the enveloping algebra operators (4.1) and base vectors L_m^S in the carrier space $\mathcal{H}_S(G)$ of irreducible representations of G coincides: the upper G-spin index S is an integral vector and the lower index ("third component") m is an integral upper-triangular matrix). Negative modes $\hat{L}_{-|m|}^I$ in (4.1) would correspond to the complex conjugate (holomorphic) vectors $L_{-m}^S \equiv \bar{L}_m^S$. We shall give later on Eq. (5.22) the explicit expression of the orbit $\mathcal{F}_0 = \{L_g \Psi_0^S, g \in G\}$ of the vacuum vector $\Psi_0^S = W_S$ under the finite left action of the group G.

Denote $\langle Sg|\Psi\rangle \equiv \Psi^S(g)$ and $\langle\Psi|Sg\rangle \equiv \bar{\Psi}^S(g)$. The coherent state overlap or "reproducing kernel" $\Delta_S(g,g') \equiv \langle Sg \mid Sg'\rangle$ can be calculated by inserting the resolution of unity

$$1 = \sum_m |\chi_m\rangle\,\langle\chi_m| \tag{5.17}$$

given by an orthonormal basis $\{|\chi_m\rangle\}$ of $\mathcal{H}_S(G)$. The explicit expression of this overlap in terms of upper-minors Δ_β, $\beta = 1,\ldots,N-1$, of $g = (t,z,\bar{z}) \in G$ turns out to be:

$$\Delta^S(g,g') = \sum_m \chi_m^S(g)\bar{\chi}_m^S(g') = \prod_{\beta=1}^{N-1} \frac{(\bar{t}_\beta|\Delta_\beta(z,\bar{z})|)^{2S_\beta}(t'_\beta|\Delta_\beta(z',\bar{z}')|)^{2S_\beta}}{|\Delta_\beta(z',\bar{z})|^{4S_\beta}}. \tag{5.18}$$

This reproducing kernel satisfies the integral equation of a projector operator

$$\Delta^S(g,g'') = \int \Delta^S(g,g')\Delta^S(g',g'')\,\mathrm{d}^L g', \tag{5.19}$$

and the propagator equation:

$$\Psi^S(g) = \int_G \mathrm{d}^L g'\,\Delta^S(g,g')\Psi^S(g'), \tag{5.20}$$

where we have used the resolution of unity

$$1 = \int_G \mathrm{d}^L g\;|Sg\rangle\,\langle Sg|. \tag{5.21}$$

Given a vector $\gamma \in \mathcal{H}_S(G)$ (for example the vacuum $W_S(g) \equiv \langle Sg|0\rangle$) the set of vectors in the orbit of γ under G, $\mathcal{F}_\gamma = \{\gamma_g = L_g\gamma,\ g \in G\}$, is called a family of covariant CS. We know from (5.15) that the Cartan (*isotropy*) subgroup $T = U(1)^{N-1}$ stabilizes the vacuum vector $\gamma = W_S$ up to multiplicative phase factors $t_\beta^{2S_\beta}$ (characters of T). Actually, the explicit expression of the family $\mathcal{F}_\gamma$ of CS for $\gamma = W_S$ turns out to be:

$$[L_g W_S](g') = W_S(g^{-1} \bullet g') = W_S(g')\,\mathrm{e}^{-\Theta_S(\bar{z}',g)} \prod_{\beta=1}^{N-1} t_\beta^{2S_\beta}, \tag{5.22}$$

where we define

$$\Theta_S(\bar{z}',g) \equiv -\sum_{\beta=1}^{N-1} 2S_\beta \ln \frac{|\Delta_\beta(z,\bar{z})|}{|\Delta_\beta(z,\bar{z}')|^2} \tag{5.23}$$

an anti-holomorphic function of $\bar{z}'$ fulfilling cocycle properties (see below) and related to the so-called "multipliers" (Radon–Nikodym derivative) in standard representation theory.

Considering the flag manifold $\mathbb{F} = G/T$ and taking the Borel section $\sigma : \mathbb{F} \to G$, $\sigma(z,\bar{z}) = (z,\bar{z},t=1) = g$ (which appears implicitly in the factorization $g = vt$) we may define another family of covariant CS as $\gamma_{\sigma(z,\bar{z})} = L_{\sigma(z,\bar{z})}\gamma$ (classes of CS modulo T), which are usually referred to as the Gilmore–Perelomov CS.

It is also known in the literature that the flag manifold $\mathbb{F}$ is a Kähler manifold, with local complex coordinates $z_{\alpha\beta}, \bar{z}_{\alpha\beta}$ (5.2), an Hermitian Riemannian metric η and a corresponding closed two-form (Kähler form) Ω,

$$ds^2 = \eta^{\alpha\beta,\mu\nu}\,dz_{\alpha\beta}\,d\bar{z}_{\mu\nu}, \qquad \Omega = i\eta^{\alpha\beta,\mu\nu}\,dz_{\alpha\beta}\wedge d\bar{z}_{\mu\nu}, \tag{5.24}$$

which can be obtained from the Kähler potential

$$K_S(z,\bar{z}) \equiv -\sum_{\beta=1}^{N-1} 4S_\beta \ln|\Delta_\beta(z,\bar{z})| \tag{5.25}$$

through the formula $\eta^{\alpha\beta,\mu\nu} = \frac{\partial}{\partial z_{\alpha\beta}}\frac{\partial}{\partial \bar{z}_{\mu\nu}}K_S$. Notice that the Kähler potential K_S essentially corresponds to the natural logarithm of the squared vacuum modulus $K_S(z,\bar{z}) = -\ln|W_S(z,\bar{z},t)|^2$ in (5.13). Actually, given the holomorphic action of G on $\mathbb{F}$,

$$(z',\bar{z}') \to g(z',\bar{z}') = \sigma^{-1}(g^{-1}\bullet(z',\bar{z}',1)), \quad g\in G,\ (z',\bar{z}')\in\mathbb{F},$$

the transformation properties of K_S are inherited from those of W_S in (5.22):

$$K_S(gz,\overline{gz}) = K_S(z,\bar{z}) + \Theta_S(\bar{z},g) + \overline{\Theta_S(\bar{z},g)}. \tag{5.26}$$

The function Θ_S verifies the cocycle condition $\Theta_S(\bar{z},g'\bullet g) = \Theta_S(\overline{gz},g') + \Theta_S(\bar{z},g)$, which results from the group property $g'(gz) = (g'\bullet g)z$.

5.3. Operator symbols on flag manifolds

Let us consider the finite left translation $[L_{g'}\Psi^S](g) \equiv \Psi^S(g'^{-1}\bullet g)$ as a linear operator in $\mathcal{H}_S(G)$. The symbol [remember the definition (2.23)] $L^S_{g'}(g,h)$, $g,g',h\in G$ of the operator $L_{g'}$ representing the group element $g'\in G$ in $\mathcal{H}_S(G)$ can be written in terms of the reproducing kernel (5.18) as:

$$L^S_{g'}(g,h) = \langle Sg|L_{g'}|Sh\rangle = \Delta^S(g'^{-1}g,h). \tag{5.27}$$

Knowing that right-invariant vector fields X^R [defined in (5.7)] are the infinitesimal generators of finite left translations L_g, one can easily compute the symbols $X^S_j(g,h)$ of the Lie-algebra $\mathcal{G}$ generators $\hat{X}_j$ as:

$$X^S_j(g,h) \equiv \langle Sg|\hat{X}_j|Sh\rangle = X^R_j(g)\Delta^S(g,h) = \mathcal{R}^k_j(g)\frac{\partial}{\partial g^k}\Delta^S(g,h). \tag{5.28}$$

From a quantum-mechanical perspective, the points $g\in G$ do not label distinct states $|g\rangle = L_g|0\rangle$ because of the inherent phase freedom in quantum mechanics. Rather, the corresponding quantum state depends on its equivalence class $(z,\bar{z}) = gT$ modulo T. Let us consider then the new action of G on the anti-holomorphic part $\Phi(\bar{z})$ of $\Psi^S(g)$ in (5.11). Since the vacuum W_S is a fixed common factor of all the wave functions $\Psi^S = W_S\Phi$ in (5.11), we can factor it out and consider the restricted action $L^S_g \equiv W_S^{-1}L_gW_S$ on the arbitrary anti-holomorphic part $\Phi(\bar{z})$, thus resulting in:

$$[L^S_g\Phi](\bar{z}') = e^{-\Theta_S(\bar{z}',g)}\Phi(g^{-1}\bar{z}'), \quad g=(z,\bar{z},t)\in G \tag{5.29}$$

(modulo T). The infinitesimal generators X_j^S of this new restricted action can be written as:

$$X_j^S = \nabla_j - \theta_j^S(\bar{z}), \tag{5.30}$$

where

$$\nabla_j \equiv X_j^R(g)(g\bar{z})_{\alpha\beta}|_{g=e}\frac{\partial}{\partial \bar{z}_{\alpha\beta}}, \qquad \theta_j^S(\bar{z}) \equiv X_j^R(g)\Theta_S(\bar{z}, g)|_{g=e}.$$

Denoting now $\langle \bar{z}|\Phi\rangle \equiv \Phi(\bar{z})$, $\langle \Phi|\bar{z}\rangle \equiv \bar{\Phi}(\bar{z})$ and $L_g^S(\bar{z}, z') \equiv \langle \bar{z}|L_g^S|\bar{z}'\rangle$, the restriction of the symbols (5.28) to the flag manifold $\mathbb{F}$ can be written in terms of the Kähler potential K_S and the cocycle Θ_S as follows:

$$X_j^S(\bar{z}, z') = X_j^R(g)L_g^S(\bar{z}, z')|_{g=e} = \nabla_j K_S(\bar{z}, z') - \theta_j^S(\bar{z}). \tag{5.31}$$

The diagonal part $X_j^S(\bar{z}, z)$ are called *equivariant momentum maps*. Using Lie equations for ∇_j and differential properties of the cocycle θ_j^S, one can prove that momentum maps implement a realization of the Lie algebra $\mathcal{G}$ of G in terms of Poisson brackets (2.17).

The correspondence between commutator (2.5) and Poisson bracket (2.17) does not hold in general for arbitrary elements like (4.1) in the universal enveloping algebra $\mathcal{U}(\mathcal{G})$. As we stated in Eq. (2.24), the star commutator of symbols admits a power series expansion in the G-spin parameters S_β (being the Poisson bracket the leading term), so that star commutators converge to Poisson brackets for large quantum numbers $S \to \infty$.

We believe that higher-order terms in the Moyal commutators (4.20) give a "taste" of these higher order corrections to the Poisson bracket in the star commutator (2.24) of symbols, which actual expression seems hard to compute.

6. Field models on flag manifolds

Before finishing, we would like to propose some interesting applications like diffeomorphism invariant field models, based on Yang–Mills theories, and non-linear sigma models on flag manifolds.

6.1. Volume-preserving diffeomorphisms and higher-extended objects

We showed in Section 3.1 that the low-energy limit of the SU(∞) Yang–Mills action (3.7), described by space-constant (vacuum configurations) SU(∞) vector potentials $X_\mu(\tau; \vartheta, \varphi) \equiv A_\mu(\tau, \vec{0}; \vartheta, \varphi)$, turns out to reproduce the dynamics of the relativistic spherical membrane $\mathbb{F}_1 = S^2$. This view can be straightforwardly extended to arbitrary flag manifolds $\mathbb{F}_{N-1} = \mathrm{SU}(N)/U(1)^{N-1}$ just replacing the Poisson bracket on the sphere (3.4) by (2.17). Actually, as it is done for sdiff(S^2) gauge invariant Yang–Mills theories in (3.7), an action functional for a sdiff$(\mathbb{F}_{N-1})$ gauge invariant Yang–Mills theory in four dimensions could be written as:

$$S = \int \mathrm{d}^4x \langle F_{\nu\gamma}|F^{\nu\gamma}\rangle, \qquad F_{\nu\gamma} = \partial_\nu A_\gamma - \partial_\gamma A_\nu + \{A_\nu, A_\gamma\}_P,$$

$$A_\nu(x;\bar{z},z) = \sum_{\{S,m\}} A^m_{\nu S}(x) L^S_m(\bar{z},z), \quad \nu,\gamma = 1,\ldots,4, \tag{6.1}$$

where now $\langle\cdot|\cdot\rangle$ denotes the scalar product between tensor operator symbols L^S_m on $\mathbb{F}_{N-1}$, with integration measure (2.18), which explicit expression is straightforwardly obtained from the left-invariant Haar measure (5.14) on the whole group G after inner derivation i_X by left-invariant generators of toral (Cartan T) elements:

$$\mathrm{d}\mu(z,\bar{z}) = \prod_{\beta=1}^{N-1} i_{X^L_{t_\beta}}\, \mathrm{d}^L g = \prod_{\beta=1}^{N-1} |\Delta_\beta(z,\bar{z})|^4 \bigwedge_{\alpha>\beta} \mathrm{d}z_{\alpha\beta} \wedge \mathrm{d}\bar{z}_{\alpha\beta}.$$

Hence, all (infinite) higher-G-spin S vector fields $A^m_{\nu S}(x)$ on $\mathbb{R}^4$ are combined into a single field $A_\nu(x;z,\bar{z})$ on the extended manifold $\mathbb{R}^4 \times \mathbb{F}_{N-1}$; that is, $A^m_{\nu S}(x)$ can be considered as a particular "vibration mode of the $N(N-1)$-brane" $\mathbb{F}_{N-1}$.

In the same way, a 2 + 1-dimensional Chern–Simons sdiff$(\mathbb{F}_{N-1})$-invariant gauge theory can be formulated with action:

$$S = \int_{\mathbb{R}^3\times\mathbb{F}_{N-1}} \left(A \wedge \mathrm{d}A + \frac{1}{3}\{A,A\} \wedge A\right), \quad A = A_\mu\, \mathrm{d}x^\mu, \tag{6.2}$$

and equations of motion: $F = 0$.

6.2. Nonlinear sigma models on flag manifolds

Let us consider a matrix $v \in \mathrm{SU}(N)/T$ (as a gauge group, i.e. as a map $v : \mathbb{R}^D \to \mathrm{SU}(N)$), which is a juxtaposition $v = (v_1,\ldots,v_N)$ of the N orthonormal vectors v_α in (5.4). The Maurer–Cartan form can be decomposed in diagonal and off-diagonal parts

$$v^{-1}\mathrm{d}v = v^\dagger\, \mathrm{d}v = \begin{pmatrix} \bar{v}^t_1 \\ \vdots \\ \bar{v}^t_N \end{pmatrix} \left(\mathrm{d}v_1,\ \cdots,\ \mathrm{d}v_N\right) = \sum_{\alpha=1}^N \bar{v}^t_\alpha\, \mathrm{d}v_\alpha \hat{X}_{\alpha\alpha} + \sum_{\alpha\neq\beta} \bar{v}^t_\alpha\, \mathrm{d}v_\beta \hat{X}_{\alpha\beta}, \tag{6.3}$$

where $\hat{X}_{\alpha\beta}$ are the step operators (2.4). The Lagrangian density for the non-linear sigma model (SM) on the coset (flag) G/T

$$\mathcal{L}_{\mathrm{SM}} = \frac{\kappa}{8} \mathrm{tr}_{G/T}(v^{-1}\partial_\mu v v^{-1}\partial_\mu v) \tag{6.4}$$

is written in terms of the off-diagonal parts as

$$\mathcal{L}_{\mathrm{SM}} = \frac{\kappa}{2} \sum_{\alpha<\beta} (v_\alpha, \partial_\mu v_\beta)^2. \tag{6.5}$$

The usual Lagrangian for the complex projective space $\mathbb{C}P^{N-1} = \mathrm{SU}(N)/(\mathrm{SU}(N-1) \times U(1))$

$$\mathcal{L}_{\mathbb{C}P^{N-1}} = \frac{\kappa}{2}\eta_{\alpha\bar{\beta}}(\varphi)\frac{\partial\varphi^\alpha}{\partial x^\mu}\frac{\partial\bar{\varphi}^\beta}{\partial x_\mu}, \quad \eta_{\alpha\bar{\beta}}(\varphi) \equiv \delta_{\alpha\beta} - \varphi_\alpha\bar{\varphi}_\beta,\ \varphi^\dagger \cdot \varphi = 1 \tag{6.6}$$

can be also obtained as a particular case of (6.5) as follows. Unitary matrices w on the coset $\mathbb{C}P^{N-1}$ are obtained from $[g]_B$ in (5.1) by considering the particular local complex parametrization where $z_{\alpha\beta} = 0 \forall \beta \geq 2$. Let us consider a new basis $\{\hat{J}^k, k = 1, \ldots, N^2 - 1\}$ of traceless Hermitian matrices for the Lie algebra su(N), normalized as $\mathrm{tr}(\hat{J}^k\hat{J}^l) = \frac{1}{2}\delta_{kl}$. Let us use $|0\rangle$ for the Dirac notation for the vacuum vector $W_S(g) = \langle Sg|0\rangle$. In the fundamental representation (lowest S), and for the $\mathbb{C}P^{N-1}$ case, the vacuum is given by the column N-vector

$$|0\rangle = \left(1,\ 0,\ \cdots,\ 0\right)^t.$$

If we define by

$$q^k \equiv \langle 0|w\hat{J}^k w^\dagger|0\rangle = (w\hat{J}^k w^\dagger)_{11}$$

the vacuum expectation value of the conjugated Lie algebra element $w\hat{J}^k w^\dagger$ under the adjoint action of the group G, then the restriction

$$\mathcal{L}_{\mathbb{C}P^{N-1}} = \frac{\kappa}{2}\sum_{\beta=2}^{N}(w_1, \partial_\mu w_\beta)^2$$

of (6.5) to $\mathbb{C}P^{N-1}$ could also be written as

$$\mathcal{L}_{\mathbb{C}P^{N-1}} = \frac{\kappa}{2}\partial_\mu q \cdot \partial^\mu q,$$

which coincides with (6.6) when we identify $\varphi_\alpha \equiv w_{\alpha 1} = z_{\alpha 1}|\Delta_1(z, \bar{z})|$, $z_{11} \equiv 1$. In particular, with this change of variable, one can see that the metric $\eta_{\alpha\bar{\beta}}$ in (6.6) coincides with $\eta^{\alpha 1;\beta 1}$ in (5.24) for the restriction $K_{S_1}(z, \bar{z}) \equiv -4S_1 \ln|\Delta_1(z, \bar{z})|$ of the Kähler potential to $\mathbb{C}P^{N-1}$.

7. Conclusions and outlook

We provided a general view of, what we agreed to call, "generalized $\mathcal{W}_\infty$ symmetries", from various perspectives and approaches. We started discussing the structure of these new infinite-dimensional $\mathcal{W}$-like Lie algebras inside a group theoretical framework as algebras of $U(N_+, N_-)$ tensor operators. Inside this context, the (hard) problem of computing commutators of tensor operators has been rephrased in terms of (more easy) Moyal brackets of (polynomial) functions on the coalgebra $u(N_+, N_-)^*$, up to quotients by the ideals generated by null-type polynomials like (4.10). That is, we have intended to recover quantum commutators from quantum (Moyal) deformations of classical (oscillator) brackets. Moyal bracket captures the essence of the full (quantum) algebra, and makes use of the standard oscillator realization of the basic $u(N_+, N_-)$-Lie algebra generators. The resulting infinite-dimensional generalized $\mathcal{W}$-algebras can be seen as:

1. infinite continuations of the finite-dimensional symmetries $u(N_+, N_-)$, or as
2. higher-$U(N_+, N_-)$-spin extensions of the diffeomorphism algebra $\mathrm{diff}(N_+, N_-)$ of a N-dimensional manifold (e.g. a N-torus).

In order to justify the view of $\mathcal{W}_\infty(N_+, N_-)$ as a "higher-spin algebra" of $U(N_+, N_-)$, we have computed higher-spin representations of $U(N_+, N_-)$ (discrete series), we have given explicit expressions for coherent states and we have derived Kähler structures on flag manifolds, which are essential ingredients to define operator symbols.

These infinite-dimensional Lie algebras potentially provide a new arena for integrable field models in higher dimensions, of which we have briefly mentioned gauge dynamics of higher-extended objects and reminded non-linear SM on flag manifolds. An exhaustive study of central extensions of $\mathcal{W}_\infty(N_+, N_-)$ should give us an important new ingredient regarding the constructions of unitary irreducible representations and invariant geometric action functionals, just as central extensions of standard $\mathcal{W}$ and Virasoro algebras encode essential information. This should be our next step.

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